Example 3 Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} dydx$

Solution:

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} dydx = \int_0^a (y)_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \sqrt{a^2-x^2}\, dx$$

$$= \left[\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2}\right]_0^a \left(\because \int \sqrt{a^2-x^2}\, dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a}\sqrt{a^2-x^2}\right)$$

$$= \frac{a^2}{2}\sin^{-1}(1)$$

$$= \frac{a^2}{2}\frac{\pi}{2} = \frac{\pi a^2}{4}$$

Example 4 Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2}$

Solution:

$$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2} = \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{dy}{\left(1+x^2\right)+y^2}\right] dx$$

$$= \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{dy}{\left(\sqrt{1+x^2}\right)^2 + y^2}\right] dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left(\tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right)_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left(\tan^{-1}\left(1\right) - \tan^{-1}\left(0\right)\right) dx \qquad \left[Q \int \frac{dx}{x^2+a^2} \tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}} dx$$

$$= \frac{\pi}{4}\left[\log\left(x+\sqrt{1+x^2}\right)\right]_0^1 \qquad \left[Q \int \frac{dx}{\sqrt{1+x^2}} = \log\left(x+\sqrt{1+x^2}\right)\right]$$

$$= \frac{\pi}{4}\left[\log\left(1+\sqrt{2}\right) - \log 1\right]$$

$$= \frac{\pi}{4}\left[\log\left(1+\sqrt{2}\right)\right]$$

4.3 EVALUATE THE DOUBLE INTEGRAL (REGION FORM)

Fubini's theorem:

If f is continuous on the rectangle
$R = \{(x, y) \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Example 1 Evaluate the double integral $\int\int_R \left(x-3y^2\right) dydx$ Where

$$R = \left\{(x,y) \mid 0 \le x \le 2,\ 1 \le y \le 2\right\},$$

Solution:

$$\int\int_R \left(x-3y^2\right) dydx = \int_0^2 \int_1^2 \left(x-3y^2\right) dydx$$

$$= \int_0^2 \left[xy - y^3\right]_1^2 dx$$

$$= \int_0^2 \left\{(2x-8)-(x-1)\right\} dx$$

$$= \int_0^2 (x-7)\, dx = \left[\frac{(x-7)^2}{2}\right]_0^2$$

$$= \frac{1}{2}\left[25-49\right] = -12.$$

Example 2 Evaluate $\int\int_R y\sin(xy)\, dydx$, $R = [1, 2] \times [0, \pi]$.

Solution:

$$\int\int_R y\sin(xy)\, dydx, = \int_0^\pi \int_1^2 y\sin(xy)\, dxdy$$

$$= \int_0^\pi \left[-y\frac{\cos(xy)}{y}\right]_1^2 dy$$

$$= \int_0^\pi \left[-\cos(xy)\right]_1^2 dy$$

$$= \int_0^\pi \left(-\cos 2y + \cos y\right) dy$$

$$= \left[\frac{-\sin 2y}{2} + \sin y\right]_0^\pi = 0.$$

Example 3 Evaluate $\iint_D (x+2y)\,dydx$, Where D is the region bounded by the parabolas $y = 2x^2$ & $y = 1 + x^2$

Solution:
The parabolas intersect when $2x^2 = 1 + x^2$, that is, $x^2 = 1$, so $x = -1$. We note that the region D, sketched in Figure 4.1, is a type I region but not a type II region and we can write

$$D = \{(x, y) \mid -1 \leqslant x \leqslant 1,\ 2x^2 \leqslant y \leqslant 1 + x^2\}$$

Since the lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$, Equation 3 gives

$$\iint_D (x + 2y)\, dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\, dy\, dx$$

$$= \int_{-1}^{1} \left[xy + y^2\right]_{y=2x^2}^{y=1+x^2} dx$$

$$= \int_{-1}^{1} [x(1 + x^2) + (1 + x^2)^2 - x(2x^2) - (2x^2)^2]\, dx$$

$$= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\, dx$$

$$= -3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x \Big]_{-1}^{1} = \frac{32}{15}$$

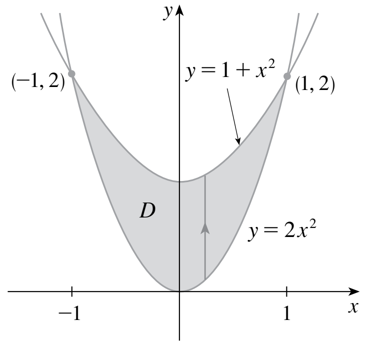


Figure 4.1

4.4 CHANGE OF ORDER OF INTEGRATION

As stated earlier, in the double integral with constant limits, the order of integration is immaterial, provided the limits of integration are changed accordingly. But in case of double integral with variable limits, the limits of the integration changes with the change in the order of the integration. The new limits are obtained by drawing a rough sketch of the region of integration. Sometimes in changing the order of integration, it is required to split up the region of integration, and the given integral is expressed as the sum of number of double integrals with the changed limits. The change of order of integration often makes the evaluation of double integrals easier.

Example 1 Change the order of integration in

$$\int_0^a \int_y^a \frac{x\,dx\,dy}{x^2 + y^2}$$, and hence evaluate the same.

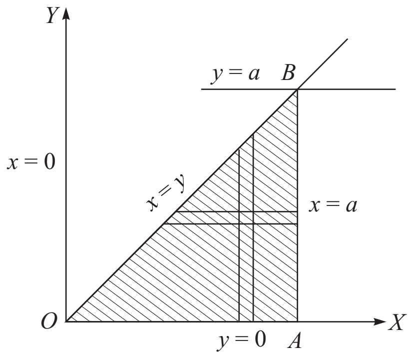


Fig. 4.2

Solution: From the limits of integration, it is clear that the region of integration is bounded by $x = y$, $x = a$, $y = 0$ and $y = a$. Thus, the region of integration is ΔOAB (see Fig. 4.2), and this region is divided into horizontal strips. To change the order of integration, divide the region of integration into vertical strips. The new limits of integration become y varies from 0 to x and x varies from 0 to a.

$$\int_0^a\int_y^a \frac{x\,dx\,dy}{x^2+y^2} = \int_0^a\int_0^x \frac{x\,dy\,dx}{x^2+y^2} = \int_0^a x\cdot\left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_0^x dx$$

$$= \int_0^a \frac{\pi}{4}\,dx = \frac{\pi}{4}\cdot[x]_0^a = \frac{\pi a}{4}$$

Example 2 Change the order of integration in the following integral and evaluate:

$$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$$

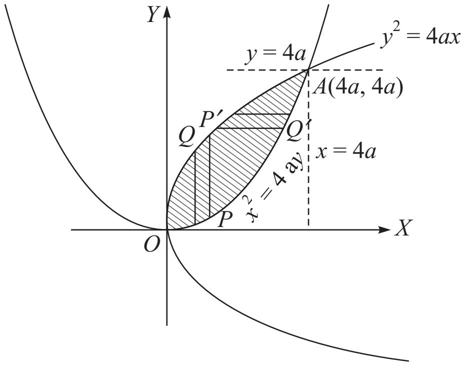


Fig. 4.3

Solution: From the limits of integration, it is clear that first the integration is to be performed w.r.t. y, which varies from $y = (x^2/4a)$ to $y = 2\sqrt{ax}$, and then w.r.t. x, which varies from $x = 0$ to $x = 4a$. Thus, we have to first integrate along the vertical strip PQ which extends from a point P on the parabola $y = x^2/(4a)$ (i.e., $x^2 = 4ay$) to the point Q on the parabola $y = 2\sqrt{ax}$ (i.e., $y^2 = 4ax$). Then the strip slides from O to A $(4a, 4a)$, the point of intersection of the two parabolas. To change the order of integration, divide the region of integration $OPAQO$ into horizontal strips $P'Q'$ which extend from P' on the parabola $y^2 = 4ax$, i.e., $x = y^2/(4a)$ to Q' on the parabola $x^2 = 4ay$, i.e., $x = 2\sqrt{ay}$. Then this strip slides from O to A $(4a, 4a)$, i.e., varies from 0 to $4a$. Therefore,

$$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx = \int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dx\,dy = \int_0^{4a} x\Big|_{y^2/4a}^{2\sqrt{ay}} dy = \int_0^{4a}\left[2\sqrt{ay} - \frac{y^2}{4a}\right]dy$$

$$= \left[2\sqrt{a}\cdot\frac{y^{3/2}}{3/2} - \frac{y^3}{12a}\right]_0^{4a}$$

$$= \frac{4}{3}\sqrt{a}\cdot(4a)^{3/2} - \frac{64a^3}{12a} = \frac{4}{3}\sqrt{a}\cdot 8a^{3/2} - \frac{16a^2}{3} = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}$$

Example 3 Express as a single integral $\int_0^{a/\sqrt{2}}\int_0^{x} x\,dy\,dx + \int_{a/\sqrt{2}}^{a}\int_0^{\sqrt{a^2-x^2}} x\,dy\,dx$ and evaluate it.

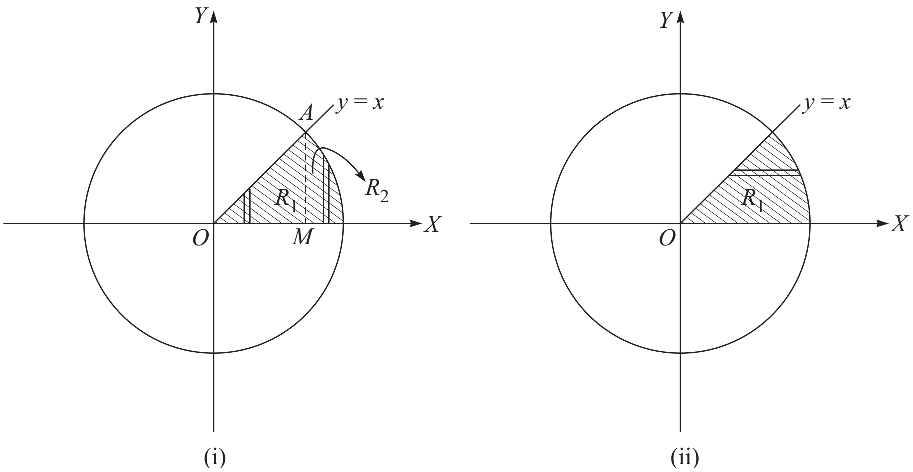


Fig. 4.4

Solution: Let $I_1 = \int_0^{a/\sqrt{2}}\int_0^{x} x\,dy\,dx$ and $I_2 = \int_{a/\sqrt{2}}^{a}\int_0^{\sqrt{a^2-x^2}} x\,dy\,dx$

Let R_1 and R_2 be the regions over which I_1 and I_2 are being integrated, respectively and are shown by the shaded region in Fig. 4.4(i).

Also from Fig. 4.4(ii), it is clear that

$$R = R_1 + R_2$$

$$\therefore \qquad I = I_1 + I_2 = \iint_R x\,dx\,dy$$

For evaluating I, change the order of integration and then take an elementary strip parallel to the x-axis from $y = x$ to $y = \sqrt{a^2 - x^2}$, i.e., the circle $x^2 + y^2 = a^2$. Thus,

$$I = \int_0^{a/\sqrt{2}}\int_y^{\sqrt{a^2-y^2}} x\,dx\,dy = \int_0^{a/\sqrt{2}}\left[\int_y^{\sqrt{a^2-y^2}} x\,dx\right]dy$$

$$= \int_0^{a/\sqrt{2}} \frac{x^2}{2}\bigg|_y^{\sqrt{a^2-y^2}} dy = \frac{1}{2}\int_0^{a/\sqrt{2}} (a^2 - y^2 - y^2)dy = \frac{1}{2}\left[a^2 y - \frac{2y^3}{3}\right]_0^{a/\sqrt{2}} = \frac{a^3}{3\sqrt{2}}$$

Example 4 Change the order of integration in $I=\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx$, and hence evaluate the same.

Solution: From the limits of integration, it is clear that the integration is to be performed first w.r.t. y which varies from $y=x^2$ to $y=2-x$ and then w.r.t. x which varies from $x=0$ to $x=1$. The shaded region in Fig. 4.5 is the region of integration. Divide this region into vertical strips. To change the order of integration, divide the region of integration into horizontal strips.

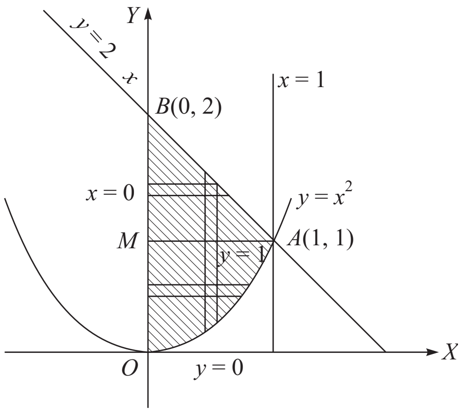


Fig. 4.5

Solving $y=x^2$ to $y=2-x$, we get the coordinates of A as (1, 1). Draw $AM \perp OY$. The region of integration is divided into two parts, OAM and MAB.

For the region OAM, x varies from 0 to $\sqrt{y}$ and y varies from 0 to 1. For the region MAB, x varies from 0 to $(2-y)$ and y varies from 1 to 2. Therefore,

$$\int_0^1\int_{x^2}^{2-x} xy\ dy\,dx=\int_0^1\int_0^{\sqrt{y}} xy\ dx\,dy+\int_1^2\int_0^{2-y} xy\ dx\,dy$$

$$=\int_0^1 y\cdot\left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy+\int_1^2 y\cdot\left[\frac{x^2}{2}\right]_0^{2-y} dy=\frac{1}{2}\int_0^1 y^2dy+\frac{1}{2}\int_1^2 y(2-y)^2\,dy$$

$$=\frac{1}{2}\cdot\left[\frac{y^3}{3}\right]_0^1+\frac{1}{2}\int_1^2(4y-4y^2+y^3)dy=\frac{1}{6}+\frac{1}{2}\left[2y^2-\frac{4y^3}{3}+\frac{y^4}{4}\right]_1^2$$

$$=\frac{1}{6}+\frac{1}{2}\left[\left(8-\frac{32}{3}+4\right)-\left(2-\frac{4}{3}+\frac{1}{4}\right)\right]=\frac{3}{8}$$

Example 5 Change the order of integration in the double integral $\int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)dy\,dx$.

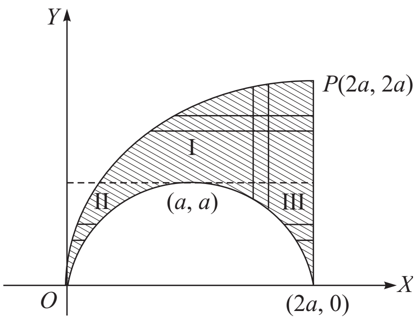


Fig. 4.6

Solution: From the limits of integration, it is clear that the integration is to be performed first w.r.t. y which varies from $y=\sqrt{2ax-x^2}$ to $y=\sqrt{2ax}$ and then w.r.t. x which varies from $x=0$ to $x=2a$. To evaluate the given integral, take the elementary strip parallel to the y-axis and its lower end is on $y=\sqrt{2ax-x^2}$, i.e., the circle $x^2+y^2-2ax=0$, and the upper end on $y=\sqrt{2ax}$, i.e., the parabola $y^2=2ax$. Then the strip is moved parallel to itself from $x=0$ to $x=2a$. Thus, the shaded portion between the parabola and the circle is the region of integration. To change the order of integration, first integrate w.r.t. x and then w.r.t. y. The elementary strip is taken parallel to the x-axis. To cover the whole shaded area the region has to be divided into following three parts, as shown in Fig. 4.6.

Region I: The strip extends from the parabola $y^2=2ax$, i.e., $x=y^2/2a$, to the straight line $x=2a$. Then the strip is taken parallel to itself from $y=a$ to $y=2a$ to cover the region I. Thus, the part of double integral in this region is given by $I_1=\int_a^{2a}\int_{y^2/2a}^{2a} f(x,y)dx\,dy$.

Region II: The strip extends from the parabola $y^2=2ax$, i.e., $x=y^2/2a$ to the circle $x^2+y^2-2ax=0$, i.e., $x=a-\sqrt{a^2-y^2}$. Then the strip is taken from $y=0$ to $y=a$ to cover the region II. Thus, the part of the integral in this region is given by $I_2=\int_0^{a}\int_{y^2/2a}^{a-\sqrt{a^2-y^2}} f(x,y)dx\,dy$.

Region III: The strip extends from the circle $x^2+y^2-2ax=0$, i.e., $x=a+\sqrt{a^2-y^2}$, to the line $x=2a$. The strip is taken from $y=0$ to $y=a$ to cover the region III. Thus, the part of the integral in this region is given by $I_3=\int_0^{a}\int_{a+\sqrt{a^2-y^2}}^{2a} f(x,\,y)dx\,dy$. Therefore,

$$\int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)dy\,dx=\int_a^{2a}\int_{y^2/2a}^{2a} f(x,y)dx\,dy+\int_0^{a}\int_{y^2/2a}^{a-\sqrt{a^2-y^2}} f(x,y)dx\,dy+\int_0^{a}\int_{a+\sqrt{a^2-y^2}}^{2a} f(x,y)dx\,dy$$

EXERCISE PROBLEMS

4.4.1. Calculate the iterated integral

(i) $\int_0^4\int_0^2\left(6x^2y-2x\right)dydx$

Ans: 222

(ii) $\int_{-3}^{3}\int_0^{\pi/2}\left(y+y^2\cos x\right)dx\,dy$

Ans: 18

(iii) $\int_1^4\int_1^2\left(x/y+y/x\right)dy\,dx$

Ans: $\frac{21}{2}$ In 2

4.4.2. Calculate the double Integral

(i) $\iint_R \sin(x-y)\,dA,\ R=\{(x,y)\,|\,0\le x\le \pi/2,\ 0\le y\le \pi/2\}$

Ans: $9/4$

(ii) $\iint_R \dfrac{xy^2}{1+x^2}\,dA,\ R=[0,\ 1]x[-3,\ 3]$

Ans: $\dfrac{1}{2}(1-\cos 1)$

(iii) $\iint_R ye^{-xy}\,dA,\ \ R=[0,\ 2]\times[0,\ 3].$

Ans: 0

4.5 EVALUATION OF DOUBLE INTEGRALS (POLAR FORM)

Introduction

To evaluate the double integral $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\,dr\,d\theta$ over the region R bounded by the curve

$r = r_1$, $r = r_2$ & the straight line $\theta = \theta_1$, $\theta = \theta_2$.

We first integrate $w{\cdot}r$ to r (keeping θ constent) between the limits r_1 & r_2 & then integrating the new expression $w{\cdot}r$ to θ between the limits θ_1 & θ_2

$$\therefore \iint_R f(r,\ \theta)\ dr d\theta = \boxed{\int_{\theta_1}^{\theta_2}\boxed{\int_{r_1}^{r_2} f(r,\theta)\,dr}\ d\theta}.$$

Example 1 Evaluate $\iint r^3\ dr\ d\theta$, over the area bounded between the circles $r = 2\cos\theta$ and $r = 4\cos\theta$.

Solution: The shaded region in Fig. 4.7 is the region of integration R. Here, r varies from $2\cos\theta$ to $4\cos\theta$ and θ varies from $-\pi/2$ to $\pi/2$. Therefore,

$$\iint_R r^3\ dr\ d\theta = \int_{-\pi/2}^{\pi/2}\int_{2\cos\theta}^{4\cos\theta} r^3 dr\ d\theta\ = \int_{-\pi/2}^{\pi/2} \left.\frac{r^4}{4}\right|_{2\cos\theta}^{4\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{4}(256\cos^4\theta - 16\cos^4)d\theta$$

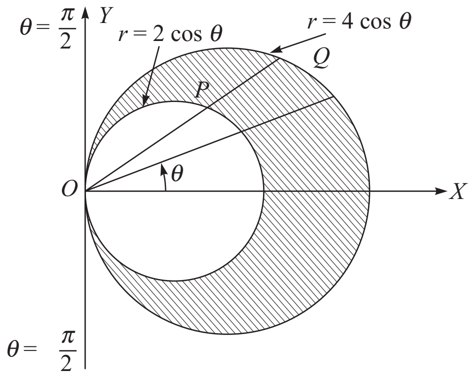


Fig. 4.7

$$= 60 \int_{-\pi/2}^{\pi/2} \cos^4\theta \, d\theta$$

$$= 120 \int_{0}^{\pi/2} \cos^4\theta \, d\theta \qquad [\because \cos^4\theta \text{ is an even function of } \theta]$$

$$= 120 \times \frac{3\times 1}{4\times 2} \times \frac{\pi}{2} = \frac{45\pi}{2}$$

Example 2 Evaluate $\iint \frac{r dr\, d\theta}{\sqrt{a^2 + r^2}}$ over one loop of the lemniscate $r^2 = a^2 \cos 2\theta$.

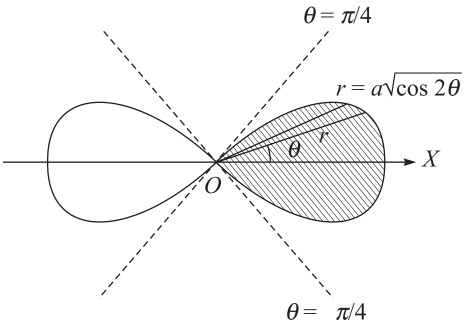


Fig. 4.8

Solution: The region of integration R is covered by radial strips whose ends are $r = 0$ and $r = a\sqrt{\cos 2\theta}$. The strips start from $\theta = -\pi/4$ and end at $\theta = \pi/4$. Therefore,

$$\iint_R \frac{r dr\, d\theta}{\sqrt{a^2 + r^2}} = \int_{-\pi/4}^{\pi/4} \int_{0}^{a\sqrt{\cos 2\theta}} \frac{1}{2}(a^2 + r^2)^{-1/2} \cdot 2r \, dr \, d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \left[\frac{1}{2} \cdot \frac{(a^2 + r^2)^{1/2}}{1/2} \right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= \int_{-\pi/4}^{\pi/4} [(a^2 + a^2 \cos 2\theta)^{1/2} - a] d\theta$$

$$= a \int_{-\pi/4}^{\pi/4} [(1 + \cos 2\theta)^{1/2} - 1] d\theta = a \int_{-\pi/4}^{\pi/4} [(2\cos^2\theta)^{1/2} - 1] d\theta$$

$$= a \int_{-\pi/4}^{\pi/4} (\sqrt{2}\cos\theta - 1) d\theta = 2a \int_{0}^{\pi/4} (\sqrt{2}\cos\theta - 1) d\theta = 2a[\sqrt{2}\sin\theta - \theta]\Big|_0^{\pi/4}$$

$$= 2a\left[\sqrt{2} \cdot \frac{1}{\sqrt{2}} - \frac{\pi}{4}\right] = 2a\left(1 - \frac{\pi}{4}\right)$$

EXERCISE

1. Evaluate the following by changing the order

(i) $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy$ ANS: $241/60$

(ii) $\int_0^1 \int_1^{\sqrt{1-x^2}} y^2 dy\,dx$ ANS: $\pi/16$

(iii) $\int_0^1 \int_x^{\sqrt{x}} xy\,dy\,dx$. ANS: $1/24$

4.6 AREA ENCLOSED BY THE PLANE CURVE

Cartesian form

Let AB and DC be two curves $y = f_1(x)$ and $y = f_2(x)$ respectively. Also Let AD and BC be the ordinates $x = x_1$ and $x = x_2$ respectively. Then the area enclosed by the curves $y = f_1(x)$ and $y = f_2(x)$
And ordinates $x=x_1$ and $x = x_2$ is ABCD.

$$\text{Area ABCD} = \int_{x_1}^{x_2} \int_{y=f_1(x)}^{y=f_2(x)} dy\,dx$$

Example 1 Show that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$ using double integration.

Solution: Solving the equations $y^2 = 4ax$ and $x^2 = 4ay$, we can see that the parabolas intersect at $O(0,0)$ and $A(4a,4a)$. For the shaded region between these parabolas (Fig. 4.26) x varies from 0 to $4a$ and y varies from P to Q, i.e. from $y = x^2/4a$ to $y = 2\sqrt{ax}$. Therefore,

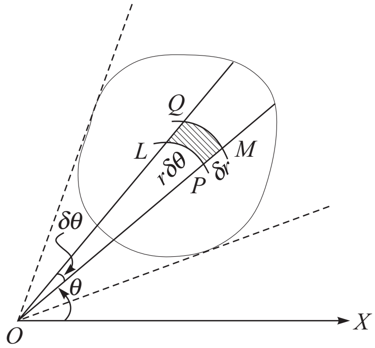


Fig. 4.9

$$\text{Required area} = \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\,dx = \int_0^{4a} (2\sqrt{ax} - x^2/4a)dx$$

$$= \left[2\sqrt{a}\cdot\frac{2}{3}x^{3/2} - \frac{1}{4a}\cdot\frac{x^3}{3} \right]_0^{4a} = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2$$

Example 2 Find the smaller of the areas bounded by the ellipse $4x^2+9y^2=36$ and the straight line $2x+3y=6$.

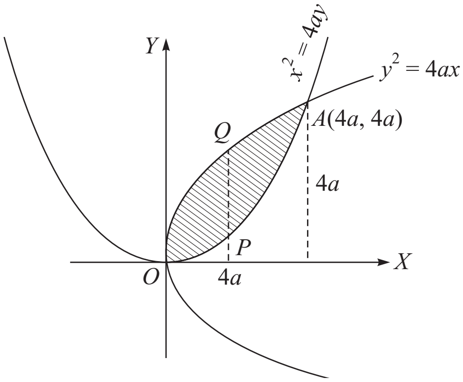


Fig. 4.10

Solution: The equation of the ellipse is $\frac{x^2}{9}+\frac{y^2}{4}=1$ (1)

and the line is $\frac{x}{3}+\frac{y}{2}=1$ (2)

Both meet the x-axis at $A(3, 0)$ and y-axis at $B(0, 2)$.
Using horizontal strips, the required area lies between

$$x=\frac{3}{2}(2-y),\ x=\frac{3}{2}\sqrt{4-y^2} \text{ and } y=0,\ y=2$$

$$\therefore \text{ Required area } =\int_0^2\int_{3/2(2-y)}^{3/2\sqrt{4-y^2}} dx\,dy=\int_0^2[x]_{3/2(2-y)}^{3/2\left(\sqrt{4-y^2}\right)} dy$$

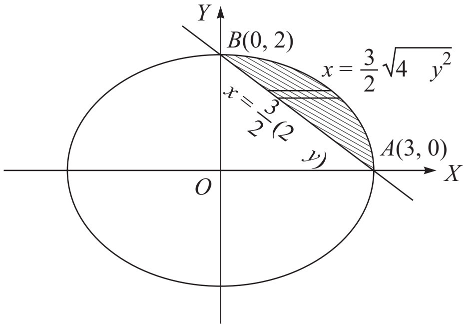


Fig. 4.11

$$=\int_0^2\frac{3}{2}\left[\sqrt{4-y^2}-(2-y)\right]dy$$

$$=\frac{3}{2}\left[\frac{y\sqrt{4-y^2}}{2}+\frac{4}{2}\sin^{-1}\frac{y}{2}-2y+\frac{y^2}{2}\right]_0^2$$

$$=\frac{3}{2}[2\sin^{-1}1-4+2]=\frac{3}{2}\left(2\cdot\frac{\pi}{2}-2\right)=\frac{3}{2}(\pi-2)$$

Example 3 Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Solution:

The area of the ellipse $=4$(area of the ellipse in the first quadrant)

$$\text{Area of the first quadrant } =\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} dy\,dx$$

$$=\int_0^a[y]_0^{b\sqrt{1-\frac{x^2}{a^2}}}dx$$

$$=\int_0^a\left[b\sqrt{1-\frac{x^2}{a^2}}\right]dx$$

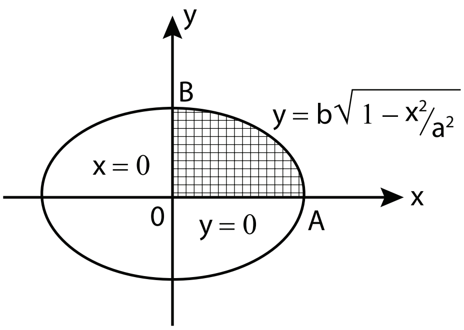


Fig. 4.12

$$= \int_0^a \left[b.\sqrt{\frac{a^2 - x^2}{a^2}} \right] dx$$

$$= \int_0^a \left[b.\sqrt{\frac{a^2 - x^2}{a^2}} \right] dx$$

$$= \frac{b}{a} \int_0^a \left[.\sqrt{a^2 - x^2} \right] dx$$

$$= \frac{b}{a} \left[\frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 - x^2} \right]_0^a \quad \left(\because \int \sqrt{a^2 - x^2}\, dx = \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 - x^2} \right)$$

$$= \frac{b}{a} \left[\frac{a^2}{2} \sin^{-1}\left(\frac{a}{a}\right) \right] = \frac{b}{a} \left[\frac{a^2}{2} \frac{\pi}{2} \right] = \left[\frac{\pi ab}{4} \right]$$

The area of the ellipse = $4\left[\frac{\pi ab}{4}\right] = \pi ab$ sq.units

Example 4 Find the area between the parabola $x^2=4y$ and the straight line $x - 2y + 4 = 0$

Solution:

Solving the equations $x^2=4y$ and $x - 2y + 4 = 0$,
Point of intersection is (4, 4) and (–2 , 1)
Limits of x: $x = -2$ and $x = 4$

Limits of y: $y = \frac{x^2}{4}$ and $y = \frac{x+4}{2}$

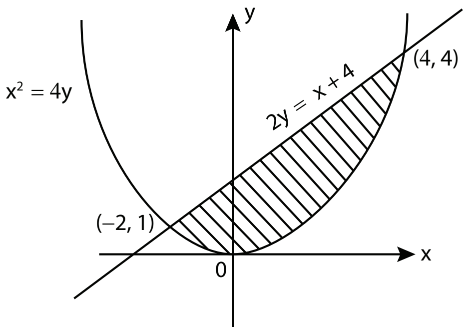


Fig. 4.13

$$\text{Area} = \int_{-2}^{4} \int_{\frac{x^2}{4}}^{\frac{x+4}{2}} dy\, dx$$

$$= \int_{-2}^{4} \left[y \right]_{\frac{x^2}{4}}^{\frac{x+4}{2}} dx$$

$$= \int_{-2}^{4} \left[\frac{x+4}{2} - \frac{x^2}{4} \right] dx$$

$$= \left[\frac{x^2}{4} + 2x - \frac{x^3}{12} \right]_{-2}^{4}$$

$= 9$ sq. units

Example 5 Find the area enclosed by the curve $y^2=4ax$ and the line $x+y=3a$, y = 0.

Solution:

To Find the intersection of $x+y=3a$, $y=0$.

Both intersecting at (3a, 0)

To find the intersection of the line $x+y=3a$ and the curve $y^2=4ax$.

Both intersecting at (a, 2a)

Given $x+y=3a$	(1)
$y^2=4ax$	(2)
We have $x+y=3a$	
$\Rightarrow$ y = 3a-x	
Substitute y = 3a-x in (2)	
We get x = a, x = 9a	
Substitute x = a in (1)	
We get y = 2a	

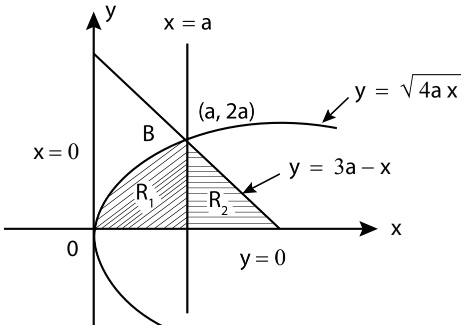


Fig. 4.14

Area split into two regions namely R1 and R2.

$$\text{Area of RI} = \int_0^a \int_0^{\sqrt{4ax}} dy\,dx$$

$$= \int_0^a (y)_0^{\sqrt{4ax}}\,dx$$

$$= \int_0^a \left(\sqrt{4ax}\right)\,dx$$

$$= 2\sqrt{a}\int_0^a x^{1/2}\,dx$$

$$= 2\sqrt{a}\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)_0^a$$

$$= \frac{4a^2}{3}$$

area of RI = 9 sq. units

$$\text{Area of RI} = \int_a^{3a} \int_0^{3a-x} dy\,dx$$

$$= \int_a^{3a} (y)_0^{3a-x}\,dx$$

$$= \int_0^a (3a-x)\,dx$$

$$= \left(\frac{(3a-x)^2}{-2}\right)_a^{3a}$$

$$= 2a^2$$

Area = Area of R1 +Area of R2

$$= \frac{4a^2}{3} + 2a^2$$

$$= \frac{10a^2}{3}$$

4.7 AREA ENCLOSED BY THE PLANE CURVE (POLAR FORM).

Example 1: Find the area enclosed by the leminiscate $r^2 = a^2 \cos 2\theta$ using double integration.

Solution: The area as shown in Fig. 4.15 is symmetrical in all the four quadrants. Therefore,

$$\text{Required area} = 4\int_0^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta = 4\int_0^{\pi/4} \frac{r^2}{2}\Bigg|_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= 2\int_0^{\pi/4} a^2 \cos 2\theta\,d\theta = 2a^2\left[\frac{1}{2}\sin 2\theta\right]_0^{\pi/4} = a^2$$

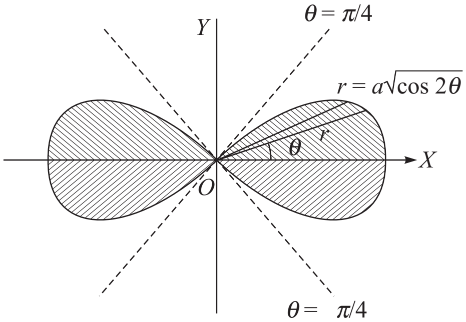


Fig. 4.15

Example 2: Find the area lying inside the circle $r = a \sin\theta$ and outside the cardioid $r = a(1 - \cos\theta)$ using double Integration.

Solution: Eliminating r between the equations of two curves, $\sin\theta = 1 - \cos\theta$ or $\sin\theta + \cos\theta = 1$

Squaring $1 + \sin 2\theta = 1$ or $\sin 2\theta = 0$

$\therefore \quad 2\theta = 0$ or π

or $\quad \theta = 0$ or $\dfrac{\pi}{2}$

For the required area, r varies from $a(1-\cos\theta)$ to $a\sin\theta$ and θ varies from 0 to $\pi/2$. Therefore,

$$\text{Required area} = \int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{r^2}{2}\Bigg|_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} a^2\left[\sin^2\theta - (1-\cos\theta)^2\right]d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} (\sin^2\theta - 1 - \cos^2\theta + 2\cos\theta)\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} (-2\cos^2\theta + 2\cos\theta)\,d\theta = a^2\left[-\frac{1}{2}.\frac{\pi}{2} + 1\right] = a^2\left(1 - \frac{\pi}{4}\right)$$

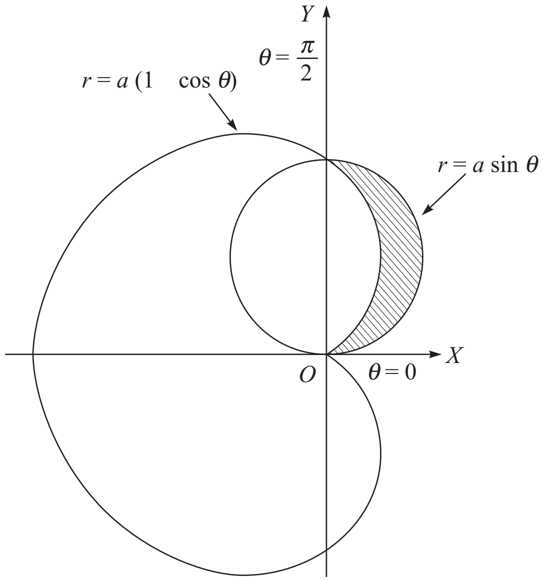


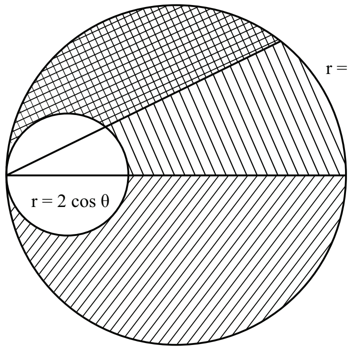
Fig. 4.16

Example 3: Find the area of the region outside the inner circle $r = 2\cos\theta$ & inside the outer circle $r = 4\cos\theta$.

Solution:

Limits of $r = 2\cos\theta$ to $4\cos\theta$

Limits of $\theta = \theta$ to $\pi/2$.



$$\text{Area} = 2\int_0^{\pi/2}\int_{2\cos\theta}^{4\cos\theta} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\left[r^2/2\right]_{2\cos\theta}^{4\cos\theta} d\theta$$

$$= \frac{2}{2}\int_0^{\pi/2}\left(16\cos^2\theta - 4\cos^2\theta\right)d\theta$$

$$= \int_0^{\pi/2} 12\cos^2\theta\, d\theta$$

$$= 12\left[\frac{1}{2}\cdot\pi/2\right]\left(\because \int_0^{\pi/2}\cos^n x dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{1}{2}\frac{\pi}{2}\right)$$

$$= 3\pi. \qquad (n-\text{even})$$

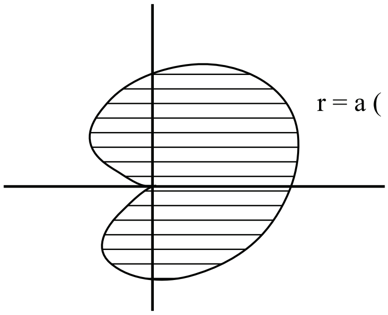
Example 4 Find the area of the cardioid $r = a(1 + \cos\theta)$

Solution:

Area = $\iint r dr d\theta$ taken over the cardioid. $r = a(1 + \cos\theta)$

limits of $r = 0$ to $r = a(1 + \cos\theta)$

limits of $\theta = -\pi$ to π



$$Area = \int_{-\pi}^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= \int_{-\pi}^{\pi}\left[r^2/2\right]_o^{a(1+\cos\theta)} d\theta$$

$$= a^2/2\int_{-\pi}^{\pi}(1+\cos\theta)^2 d\theta$$

$$= \frac{a^2}{2}\cdot 2\int_0^{\pi}\left(2\cos^2\theta/2\right)^2 d\theta \qquad \left(\because 1+\cos\theta = 2\cos^2\theta/2\right)$$

$$= a^2\int_0^{\pi} 4\cos^4\theta/2\, d\theta$$

$$= 4a^2\int_0^{\pi/2}\cos^4 y(2\,dy)$$

$$= a^2\int_0^{\pi/2}\cos^4 y\ \ dy$$