2.1.1 Partial Derivatives of First Order

Consider a function $z = f(x, y)$ of two independent variables x and y. By keeping y as a constant and varying x only, z becomes a function of x alone. The derivative of z with respect to x (y is kept constant) is called the partial derivative of z with respect to x and is denoted by $\partial z/\partial x$ or $\partial f/\partial x$ or f_x or $D_x f$. Thus,

$$\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$$

Similarly, the derivative of z with respect to y (when x is kept constant) is called the partial derivative of z with respect to y and is denoted by $\partial z/\partial y$ or $\partial f/\partial y$ or f_y or $D_y f$. Thus,

$$\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y}$$

$\partial z/\partial x$ and $\partial z/\partial y$ are called the **first order partial derivatives of z**.

In general, if z is a function of more than two independent variables, then the partial derivative of z with respect to any one of the variables, keeping all other variables constant, is the partial derivative of z with respect to that variable.

2.1.2 Partial Derivatives of Higher Order

The first order partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ being the functions of x and y can be further differentiated partially with respect to x and y to get the second order partial derivatives. The second order partial derivatives are

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \text{ or } f_{xx}, \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} \text{ or } f_{yy}, \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} \text{ or } f_{xy}, \quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} \text{ or } f_{yx}$$

The derivatives f_{xy} and f_{yx} are called mixed derivatives. In general, $\partial^2 z/\partial y\partial x = \partial^2 z/\partial x\partial y$ or $f_{xy} = f_{yx}$ if f_{xy} and f_{yx} are continuous, i.e., the order of differentiation is immaterial if the partial derivatives involved are continuous.

In general, the first order partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ can be differentiated successively to the partial derivatives of higher order.

Note:

(i) If $z = u + v$, where $u = f(x, y), v = \phi(x, y)$, then z is a function of x and y.

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

(ii) If $z = uv$, where $u = f(x, y)$, $v = \phi(x, y)$, then $\frac{\partial z}{\partial x} = \frac{\partial (uv)}{\partial x} = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}, \ \frac{\partial z}{\partial y} = \frac{\partial (uv)}{\partial y} = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}$.

(iii) If $z = u/v$, where $u = f(x, y)$, $v = \phi(x, y)$, then $\frac{\partial z}{\partial x} = \frac{\partial(u/v)}{\partial x} = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$, $\frac{\partial z}{\partial y} = \frac{\partial(u/v)}{\partial y} = \frac{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}}{v^2}$.

(iv) If $z = f(u)$, where $u = \phi(x, y)$, then $\frac{\partial z}{\partial x} = \frac{dz}{du}\cdot\frac{\partial u}{\partial x}$, $\frac{\partial z}{\partial y} = \frac{dz}{du}\cdot\frac{\partial u}{\partial y}$.

Example 2.1 Find all the partial derivatives of $ax^2 + 2hxy + by$.

Solution: Let $f(x, y) = ax^2 + 2hxy + by^2$

Then $$\frac{\partial f}{\partial x} = 2ax + 2hy, \frac{\partial f}{\partial y} = 2hx + 2by$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2ax + 2hy) = 2a$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2ax + 2hy) = 2h$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(2hx + 2by) = 2h$$

and $$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(2hx + 2by) = 2b$$

All the partial derivatives of order higher than two vanish. It may be observed that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Example 2.2 If $z = f(x + ct) + \varphi(x - ct)$, prove that $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$.

Solution: We have $$\frac{\partial z}{\partial x} = f'(x + ct)\cdot\frac{\partial}{\partial x}(x + ct) + \varphi'(x - ct)\frac{\partial}{\partial x}(x - ct) = f'(x + ct) + \varphi'(x - ct)$$

and $$\frac{\partial^2 z}{\partial x^2} = f''(x + ct) + \varphi''(x - ct) \tag{1}$$

Again $$\frac{\partial z}{\partial t} = f'(x + ct)\frac{\partial}{\partial t}(x + ct) + \varphi'(x - ct)\frac{\partial}{\partial t}(x - ct) = cf'(x + ct) - c\varphi'(x - ct)$$

and $$\frac{\partial^2 z}{\partial t^2} = c^2 f''(x + ct) + c^2\varphi''(x - ct) = c^2\left[f''(x + ct) + \varphi''(x - ct)\right] \tag{2}$$

From (1) and (2), we get $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$

Example 2.3 If $v = (x^2 + y^2 + z^2)^{-1/2}$, prove that $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=0.$

Solution: We have $\frac{\partial v}{\partial x}=-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot\ 2x=-x(x^2+y^2+z^2)^{-3/2}$

and $$\frac{\partial^2 v}{\partial x^2}=-[1\cdot(x^2+y^2+z^2)^{-3/2}+x\left(\frac{-3}{2}\right)(x^2+y^2+z^2)^{-5/2}\cdot 2x]$$

$$=-(x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2-3x^2)=(x^2+y^2+z^2)^{-5/2}(2x^2-y^2-z^2)$$

Similarly, $\frac{\partial^2 v}{\partial y^2}=(x^2+y^2+z^2)^{-5/2}(-x^2+2y^2-z^2)$ and $\frac{\partial^2 v}{\partial z^2}=(x^2+y^2+z^2)^{-5/2}(-x^2-y^2+2z^2)$

Hence, $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=(x^2+y^2+z^2)^{-5/2}\cdot(0)=0$

Note: The given equation is known as the **Laplace equation**, and the function which satisfies this equation is called the **harmonic function.**

Example 2.4 If $\theta=t^n e^{-\frac{r^2}{4t}}$, find the value of n which will make $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{\partial\theta}{\partial t}$.

Solution: Let $\theta=t^n e^{-\frac{r^2}{4t}}$

$$\frac{\partial\theta}{\partial r}=t^n\cdot e^{-\frac{r^2}{4t}}\cdot\left(\frac{-2r}{4t}\right)=-\frac{1}{2}rt^{n-1}e^{-\frac{r^2}{4t}}$$

$\therefore$ $$r^2\frac{\partial\theta}{\partial r}=-\frac{1}{2}r^3\cdot t^{n-1}e^{-\frac{r^2}{4t}}$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=-\frac{1}{2}t^{n-1}\left[3r^2e^{-\frac{r^2}{4t}}+r^3e^{-\frac{r^2}{4t}}\left(-\frac{2r}{4t}\right)\right]=-\frac{1}{2}t^{n-1}r^2e^{-\frac{r^2}{4t}}\left[3-\frac{r^2}{2t}\right]$$

$\therefore$ $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{1}{2}t^{n-1}e^{-\frac{r^2}{4t}}\left(\frac{r^2}{2t}-3\right)$$

Also, $$\frac{\partial\theta}{\partial t}=nt^{n-1}e^{-\frac{r^2}{4t}}+t^ne^{-\frac{r^2}{4t}}\cdot\left(\frac{r^2}{4t^2}\right)=t^{n-1}e^{-\frac{r^2}{4t}}\left(n+\frac{r^2}{4t}\right)$$

Since $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial t}\right)=\frac{\partial\theta}{\partial t}$$ [Given]

or $$\frac{1}{2}t^{n-1}e^{-\frac{r^2}{4t}}\left(\frac{r^2}{2t}-3\right)=t^{n-1}e^{-\frac{r^2}{4t}}\left(n+\frac{r^2}{4t}\right)$$

or $$\frac{r^2}{4t}-\frac{3}{2}=n+\frac{r^2}{4t}$$

or $$n=-\frac{3}{2}$$

Example 2.5 If $u = (1 - 2xy + y^2)^{-1/2}$, prove that $\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{y^2\frac{\partial u}{\partial y}\right\}=0$.

Solution: $u = (1 - 2xy + y^2)^{-1/2} = V^{-1/2}$, where $V = 1 - 2xy + y^2$

$$\frac{\partial u}{\partial x}=-\frac{1}{2}V^{-3/2}\cdot\frac{\partial V}{\partial x}=-\frac{1}{2}V^{-3/2}(-2y)=yV^{-3/2}$$

$$\frac{\partial^2 u}{\partial x^2}=y\cdot\frac{\partial}{\partial x}(V^{-3/2})=y\cdot\left(-\frac{3}{2}\right)V^{-5/2}\cdot\frac{\partial V}{\partial x}=-\frac{3}{2}yV^{-5/2}(-2y)=3y^2V^{-5/2}$$

$$\therefore \quad \frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\}=(1-x^2)\cdot\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\cdot\frac{\partial}{\partial x}(1-x^2)$$

$$=(1-x^2)\cdot 3y^2V^{-5/2}+yV^{-3/2}(-2x)=yV^{-3/2}[3yV^{-1}(1-x^2)-2x] \quad (1)$$

Similarly, $\frac{\partial u}{\partial y}=-\frac{1}{2}V^{-3/2}\frac{\partial V}{\partial y}=-\frac{1}{2}V^{-3/2}(-2x+2y)=V^{-3/2}\cdot(x-y)$

$$\frac{\partial^2 u}{\partial y^2}=V^{-3/2}\cdot\frac{\partial}{\partial y}(x-y)+(x-y)\cdot\frac{\partial}{\partial y}(V^{-3/2})$$

$$=V^{-3/2}\cdot(-1)+(x-y)\cdot\left(-\frac{3}{2}V^{-5/2}\right)\cdot\frac{\partial V}{\partial y}$$

$$=-V^{-3/2}-\frac{3}{2}(x-y)V^{-5/2}\cdot(-2x+2y)=-V^{-3/2}+3(x-y)^2V^{-5/2}$$

$$\therefore \quad \frac{\partial}{\partial y}\left\{y^2\frac{\partial u}{\partial y}\right\}=y^2\cdot\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}\cdot\frac{\partial}{\partial y}(y^2)$$

$$=y^2[-V^{-3/2}+3(x-y)^2V^{-5/2}]+V^{-3/2}(x-y)\cdot 2y$$

$$=yV^{-3/2}[-y+3y(x-y)^2V^{-1}+2(x-y)]$$

$$=yV^{-3/2}[3y(x-y)^2V^{-1}+(2x-3y)] \quad (2)$$

Adding (1) and (2), we have

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{y^2\frac{\partial u}{\partial y}\right\}=yV^{-3/2}[3yV^{-1}(1-x^2)-2x+3y(x-y)^2V^{-1}+2x-3y]$$

$$=yV^{-3/2}[3yV^{-1}(1-x^2+x^2-2xy+y^2)-3y]$$

$$=yV^{-3/2}[3yV^{-1}(1-2xy+y^2)-3y]$$

$$= yV^{-3/2}(3y - 3y) \qquad (\because \quad V = 1 - 2xy + y^2)$$
$$= 0$$

Example 2.6 If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, prove that $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right)$

Solution: We have $x^2(a^2 + u)^{-1} + y^2(b^2 + u)^{-1} + z^2(c^2 + u)^{-1} = 1$ (1)

Differentiating (1) partially w.r.t. x, we get

$$2x(a^2+u)^{-1} - x^2(a^2+u)^{-2}\frac{\partial u}{\partial x} - y^2(b^2+u)^{-2}\frac{\partial u}{\partial x} - z^2(c^2+u)^{-2}\frac{\partial u}{\partial x} = 0$$

or $$\frac{2x}{a^2+u} = \left\{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right\}\frac{\partial u}{\partial x}$$

or $$\frac{\partial u}{\partial x} = \frac{2x}{(a^2+u)v}, \text{ where } v = \frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}$$

Similarly, differentiating (1) partially w.r.t. y, we get

$$\frac{2y}{b^2+u} = \left\{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right\}\frac{\partial u}{\partial y} \text{ or } \frac{\partial u}{\partial y} = \frac{2y}{(b^2+u)v}$$

Similarly, differentiating (1) partially w.r.t. z, we get

$$\frac{2z}{(c^2+u)} = \left\{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right\}\frac{\partial u}{\partial z} \text{ or } \frac{\partial u}{\partial z} = \frac{2z}{(c^2+u)v}$$

$\therefore$ $$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = \frac{4}{v^2}\left\{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right\} = \frac{4}{v} \qquad (2)$$

Also, $$2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right) = 2\left\{\frac{2x^2}{(a^2+u)v} + \frac{2y^2}{(b^2+u)v} + \frac{2z^2}{(c^2+u)v}\right\}$$

$$= \frac{4}{v}\left(\frac{x^2}{(a^2+u)} + \frac{y^2}{(b^2+u)} + \frac{z^2}{(c^2+u)}\right) = \frac{4}{v} \qquad \text{[by (1)] (3)}$$

From Eqs. (2) and (3), we get

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right)$$

Example 2.7 If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that

(i) $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$$

(ii) $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+2\frac{\partial^2 u}{\partial y \partial z}+2\frac{\partial^2 u}{\partial z \partial x}+2\frac{\partial^2 u}{\partial x \partial y}=\frac{-9}{(x+y+z)^2}$$

Solution: (i) $u = \log(x^3 + y^3 + z^3 - 3xyz)$

$$\frac{\partial u}{\partial x}=\frac{3x^2-3yz}{x^3+y^3+z^3-3xyz};\ \frac{\partial u}{\partial y}=\frac{3y^2-3zx}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial z}=\frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

Adding, $$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}=\frac{3}{x+y+z}$$

$$[\because\ x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-zx)]$$

Now, $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$=-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}=-\frac{9}{(x+y+z)^2} \qquad (1)$$

(ii) $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)+\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x \partial y}+\frac{\partial^2 u}{\partial x \partial z}+\frac{\partial^2 u}{\partial y \partial x}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial y \partial z}+\frac{\partial^2 u}{\partial z \partial x}+\frac{\partial^2 u}{\partial z \partial y}+\frac{\partial^2 u}{\partial z^2}$$

$$=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+2\frac{\partial^2 u}{\partial y \partial x}+2\frac{\partial^2 u}{\partial y \partial z}+2\frac{\partial^2 u}{\partial z \partial x}$$

$$\left[\because \frac{\partial^2 u}{\partial y \partial x}=\frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial z \partial y}=\frac{\partial^2 u}{\partial y \partial z}, \frac{\partial^2 u}{\partial x \partial z}=\frac{\partial^2 u}{\partial z \partial x}\right]$$

$\therefore$ $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+2\frac{\partial^2 u}{\partial y \partial z}+2\frac{\partial^2 u}{\partial z \partial x}+2\frac{\partial^2 u}{\partial x \partial y}=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\frac{-9}{(x+y+z)^2} \qquad \text{[From (1)]}$$

2.2 HOMOGENEOUS FUNCTIONS AND EULER'S THEOREM

An expression of the form $a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_ny^n$ in which each term is of nth degree is called a **homogeneous function** of degree n in the variables x and y. It can be expressed as

$$x^n\left[a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + \cdots + a_n\left(\frac{y}{x}\right)^n\right] \text{ or } y^n\left[a_0\left(\frac{x}{y}\right)^n + a_1\left(\frac{x}{y}\right)^{n-1} + a_2\left(\frac{x}{y}\right)^{n-2} + \cdots + a_n\right]$$

Hence, any function $f(x, y)$ which can be expressed in the form $x^n\phi\left(\frac{y}{x}\right)$ or $y^n\phi\left(\frac{x}{y}\right)$ is called a homogeneous function of degree n in x and y. For example,

$$f(x,y) = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x(1+y/x)}{\sqrt{x}(1+\sqrt{y/x})} = x^{1/2}\phi\left(\frac{y}{x}\right)$$

which means $f(x, y)$ is a homogeneous function of degree 1/2 in x and y. Also,

$$f(x,y) = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{y(x/y+1)}{\sqrt{y}(\sqrt{x/y}+1)} = y^{1/2}\phi\left(\frac{x}{y}\right)$$

In general, a function $f(x, y, t, \ldots)$ is a homogeneous function of degree n in variables $x, y, t, \ldots$ if

$$f(x,y,t,\ldots) = x^n\phi\left(\frac{y}{x},\frac{t}{x},\cdots\right) \text{ or } y^n\phi\left(\frac{x}{y},\frac{t}{y},\cdots\right) \text{ or } t^n\phi\left(\frac{x}{t},\frac{y}{t},\cdots\right), \text{ etc.}$$

2.2.1 Euler's Theorem on Homogeneous Functions

Theorem 2.1 *If z is a homogeneous function of degree n in x and y, then* $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = nz$.

Proof: Since z is a homogeneous function of degree n in x and y,

$$z = x^n f\left(\frac{y}{x}\right)$$

or $$\frac{\partial z}{\partial x} = nx^{n-1}f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\cdot y\cdot\left(\frac{-1}{x^2}\right) = nx^{n-1}f\left(\frac{y}{x}\right) - yx^{n-2}f'\left(\frac{y}{x}\right)$$

or $$x\frac{\partial z}{\partial x} = nx^n f\left(\frac{y}{x}\right) - yx^{n-1}f'\left(\frac{y}{x}\right) \tag{2.9}$$

Also, $$\frac{\partial z}{\partial y} = x^n f'\left(\frac{y}{x}\right)\cdot\left(\frac{1}{x}\right) = x^{n-1}f'\left(\frac{y}{x}\right) \quad \text{or} \quad y\frac{\partial z}{\partial y} = x^{n-1}yf'\left(\frac{y}{x}\right) \tag{2.10}$$

Adding (2.9) and (2.10), we get $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = nx^n f\left(\dfrac{y}{x}\right) = nz$.

Note: In general, if z is a homogeneous function of degree n in $x, y, t, \ldots$, then $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} + t\dfrac{\partial z}{\partial t} + \cdots = nz$.

Deduction of Euler's Theorem

Theorem 2.2 *If z is a homogeneous function of degree n in x and y, then*

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)z$$

Proof: Since z is a homogeneous function of degree n in x and y, thus by Euler's theorem, we have

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz \tag{2.11}$$

Differentiating (2.11) partially w.r.t. x, we get $\frac{\partial z}{\partial x}+x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial x\partial y}=n\frac{\partial z}{\partial x}$ (2.12)

Differentiating (2.11) partially w.r.t. y, we get $\frac{\partial z}{\partial y}+x\frac{\partial^2 z}{\partial y\partial x}+y\frac{\partial^2 z}{\partial y^2}=n\frac{\partial z}{\partial y}$

But $\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial^2 z}{\partial x\partial y}$

Hence, $$\frac{\partial z}{\partial y}+x\frac{\partial^2 z}{\partial x\partial y}+y\frac{\partial^2 z}{\partial y^2}=n\frac{\partial z}{\partial y} \tag{2.13}$$

Multiplying (2.12) by x, (2.13) by y, and then adding, we get

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}+x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=n\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)$$

or $$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}+nz=n(nz)$$ [From (2.11)]

or $$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)z$$

Example 2.8 If $u=\tan^{-1}\frac{x^3+y^3}{x-y}$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$.

Solution: Here u is not a homogeneous function but $\tan u=\frac{x^3+y^3}{x-y}=\frac{x^3\left[1+(y/x)^3\right]}{x[1-(y/x)]}=x^2 f\left(\frac{y}{x}\right)$ is a homogeneous function of degree 2 in x and y.

Therefore, by Euler's theorem, we have

$$x\frac{\partial}{\partial x}(\tan u)+y\frac{\partial}{\partial y}(\tan u)=2\tan u$$

or $$x\sec^2 u\frac{\partial u}{\partial x}+y\sec^2 u\frac{\partial u}{\partial y}=2\tan u$$

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{2\sin u}{\cos u}\cdot\cos^2 u=2\sin u\cos u=\sin 2u$$

Example 2.9 If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$.

Solution: Here, $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ is not a homogeneous function but $\cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$ is homogeneous in x and y of degree 1/2.

Therefore, by Euler's theorem, we have

$$x\frac{\partial}{\partial x}\cos u + y\frac{\partial}{\partial y}\cos u = \frac{1}{2}\cos u$$

or $$-x\sin u\frac{\partial u}{\partial x} - y\sin u\frac{\partial u}{\partial y} = \frac{1}{2}\cos u$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$$. Hence the result.

Example 2.10 If $u = \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$$

and $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin u\cos 2u}{4\cos^3 u}$$

Solution: Here u is not a homogeneous function but $z = \sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$ is a homogeneous function of degree 1/2 in x and y. Therefore,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{2}z$$

or $$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{1}{2}\sin u$$

Thus, $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u \qquad (1)$$

Differentiating (1) partially w.r.t. x, we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\partial y} = \frac{1}{2}\sec^2 u\frac{\partial u}{\partial x}$$

or $$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = \left(\frac{1}{2}\sec^2 u - 1\right)\frac{\partial u}{\partial x} \qquad (2)$$

Again differentiating (1) partially w.r.t. y, we get

$$x\frac{\partial^2 u}{\partial y\partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = \frac{1}{2}\sec^2 u\frac{\partial u}{\partial y}$$

or $$x\frac{\partial^2 u}{\partial x \partial y}+y\frac{\partial^2 u}{\partial y^2}=\left(\frac{1}{2}\sec^2 u-1\right)\frac{\partial u}{\partial y} \qquad (3)$$

Multiplying (2) by x and (3) by y and adding, we obtain

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x \partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left(\frac{1}{2}\sec^2 u-1\right)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

or $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x \partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left(\frac{1}{2}\sec^2 u-1\right)\left(\frac{1}{2}\tan u\right) \qquad \text{[by (1)]}$$

$$=\frac{1}{4}\frac{\sin u}{\cos^3 u}-\frac{1}{2}\frac{\sin u}{\cos u}$$

$$=-\frac{\sin u(2\cos^2 u-1)}{4\cos^3 u}$$

Hence, $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x \partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\frac{\sin u\cos 2u}{4\cos^3 u}$$

2.3 TOTAL DERIVATIVE AND PARTIAL DIFFERENTIATION OF IMPLICIT FUNCTIONS

If $z=f(x,y)$, where $x=\phi(t),\ y=\psi(t)$, then z is called a **composite function of the single variable** t. We can substitute the values of x and y in $f(x, y)$ and express z as a function of t alone. We can then obtain dz/dt, which is called the **total derivative**. However, the substitution of $x=\phi(t),\ y=\psi(t)$ is not always convenient. Then the use of partial derivatives helps in finding the total derivative. This method is given in the next section.

Similarly, if $z=f(x,y)$, where $x=\phi(u,v),\ y=\psi(u,v)$, then z is called a **composite function of the two variables** u and v and we can find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

2.3.1 Chain Rule for Partial Differentiation

Theorem 2.3 If $z=f(x,y)$ *where* $x=\phi(t),\ y=\psi(t)$, i.e., z is a composite function of t, then

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt} \qquad (2.14)$$

Proof: We have $z=f(x,y)$, where $x=\phi(t),\ y=\psi(t)$ (2.15)

Let an increment to t be δt and the corresponding increments to x, y and z be δx, δy and δz, respectively. Then

$$z+\delta z=f(x+\delta x, y+\delta y) \qquad (2.16)$$

Subtracting (2.15) from (2.16), we get

$$\delta z = f(x+\delta x, y+\delta y) - f(x,y)$$

$$= \{f(x+\delta x, y+\delta y) - f(x, y+\delta y)\} + \{f(x, y+\delta y) - f(x,y)\}$$

Hence, $$\frac{\delta z}{\delta t} = \frac{\{f(x+\delta x, y+\delta y) - f(x, y+\delta y)\}}{\delta x}\cdot\frac{\delta x}{\delta t} + \frac{\{f(x, y+\delta y) - f(x,y)\}}{\delta y}\cdot\frac{\delta y}{\delta t}$$

Taking limit as $\delta t \to 0$, δx and δy also $\to 0$, we have

$$\lim_{\delta t\to 0}\frac{\delta z}{\delta t} = \lim_{\substack{\delta x\to 0\\ \delta y\to 0}}\left\{\frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x}\right\}\lim_{\delta t\to 0}\frac{\delta x}{\delta t} + \lim_{\delta y\to 0}\left\{\frac{f(x, y+\delta y) - f(x,y)}{\delta y}\right\}\lim_{\delta t\to 0}\frac{\delta y}{\delta t}$$

or $$\frac{dz}{dt} = \lim_{\delta y\to 0}\left[\frac{\partial f(x, y+\delta y)}{\partial x}\right]\cdot\frac{dx}{dt} + \frac{\partial f(x,y)}{\partial y}\cdot\frac{dy}{dt}$$ [Assuming $\partial f(x,y)/\partial x$ as a continuous function of y]

$$= \frac{\partial f(x,y)}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f(x,y)}{\partial y}\cdot\frac{dy}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

which is known as the **chain rule**.

Note:

(i) Taking $t = x$, (2.14) becomes $$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dx} \quad (2.17)$$

(ii) If $z = f(x, y, s, ...)$, where $x, y, s, \dots$ are all functions of a variable t, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial z}{\partial s}\cdot\frac{ds}{dt} + \cdots$$

(iii) If $z = f(x,y)$ and x, y are functions of u and v, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}, \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

2.3.2 Differentiation of Implicit Functions

If $f(x,y) =$ constant, c (say) is an implicit relation between x and y, i.e., y is an implicit function of x, then

$$\frac{df}{dx} = 0 \quad \text{or} \quad \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} = 0$$ [Using (2.17)]

This implies that $$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{f_x}{f_y} \quad \left(\frac{\partial f}{\partial y} \neq 0\right) \quad (2.18)$$

which is the first differential coefficient of an implicit function.

We can also express $\frac{d^2 y}{dx^2}$ in terms of partial derivatives. Let $\frac{\partial f}{\partial x} = p, \frac{\partial f}{\partial y} = q, \frac{\partial^2 f}{\partial x^2} = r, \frac{\partial^2 f}{\partial y^2} = t, \frac{\partial^2 f}{\partial x \partial y} = s$.

$$\frac{dy}{dx} = -\frac{p}{q} \qquad \text{[From (2.18)]}$$

On differentiating again w.r.t. x, we obtain

$$\frac{d^2y}{dx^2} = -\frac{q(dp/dx) - p(dq/dx)}{q^2}. \tag{2.19}$$

But by chain rule, $\frac{dp}{dx} = \frac{\partial p}{\partial x}\cdot 1 + \frac{\partial p}{\partial y}\cdot\frac{dy}{dx} = r + s\left(\frac{-p}{q}\right) = \frac{qr - ps}{q}$

and $\frac{dq}{dx} = \frac{\partial q}{\partial x}\cdot 1 + \frac{\partial q}{\partial y}\cdot\frac{dy}{dx} = s + t\left(\frac{-p}{q}\right) = \frac{qs - pt}{q}$

Substituting the values of dp/dx and dq/dx in (2.19), we get

$$\frac{d^2y}{dx^2} = -\frac{1}{q^2}\left[q\left(\frac{qr-ps}{q}\right) - p\left(\frac{qs-pt}{q}\right)\right] = -\frac{1}{q^3}(q^2r - 2pqs + p^2t)$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{q^2r - 2pqs + p^2t}{q^3}$$

which is the second differential coefficient of an implicit function.

Example 2.11 If $u = \frac{x^3y^3z^3}{x^3+y^3+z^3} + \log\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right)$, then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

Solution: Let $v = \frac{x^3y^3z^3}{x^3+y^3+z^3}$ and $w = \log\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right)$. (1)

so that $u = v + w$

Since $v = x^6\,\frac{(y/x)^3(z/x)^3}{1+(y/x)^3(z/x)^3}$, v is a homogeneous function of degree 6 in x, y, z.

Hence, by Euler's theorem, we have

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = 6v \tag{2}$$

Since $w = \log\left\{\frac{\frac{y}{x} + \frac{y}{x}\cdot\frac{z}{x} + \frac{z}{x}}{1 + \left(\frac{y}{x}\right)^2 + \left(\frac{z}{x}\right)^2}\right\}$, therefore w is a homogeneous function of degree zero in x, y, z.

Hence, by Euler's theorem, we have

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = 0 \tag{3}$$

Adding (2) and (3), we obtain

$$x\left(\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\right)+y\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial y}\right)+z\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial z}\right)=6v$$

or
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=6\ \frac{x^3y^3z^3}{x^3+y^3+z^3}$$

Example 2.12 If z is a function of x and y, where $x=e^u+e^{-v}$ and $y=e^{-u}-e^v$, show that

$$\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}$$

Solution: Here z is a composite function of u and v. Therefore,

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}=\frac{\partial z}{\partial x}\cdot e^u+\frac{\partial z}{\partial y}(-e^{-u})$$

and
$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}=\frac{\partial z}{\partial x}(-e^{-v})+\frac{\partial z}{\partial y}(-e^{v})$$

Subtracting,
$$\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=(e^u+e^{-v})\frac{\partial z}{\partial x}-(e^{-u}-e^v)\frac{\partial z}{\partial y}=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}$$

Example 2.13 (i) If $y^x+x^y=(x+y)^{(x+y)}$, find $\frac{dy}{dx}$.

(ii) If $z=\sqrt{x^2+y^2}$ and $x^3+y^3+3axy=5a^3$, find the value of $\frac{dz}{dx}$ when $x=y=a$.

Solution: (i) Let $f(x,y)=y^x+x^y-(x+y)^{(x+y)}$

Now,
$$\frac{\partial f}{\partial x}=y^x\log y+y\cdot x^{y-1}-(x+y)^{x+y}[1+\log(x+y)]$$

and
$$\frac{\partial f}{\partial y}=x\,y^{x-1}+x^y\log x-(x+y)^{x+y}[1+\log(x+y)]$$

Hence,
$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}=-\left[\frac{y^x\log y+y\,x^{y-1}-(x+y)^{x+y}\left(1+\log(x+y)\right)}{x\,y^{x-1}+x^y\log x-(x+y)^{x+y}\left(1+\log(x+y)\right)}\right]$$

(ii) Since $z=\sqrt{x^2+y^2}$, thus by the concept of total derivative, we have

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$$

$$\Rightarrow\quad \frac{dz}{dx}=\frac{\partial z}{\partial x}\cdot 1+\frac{\partial z}{\partial y}\frac{dy}{dx}$$

$$=\frac{1}{2}(x^2+y^2)^{-1/2}2x\cdot 1+\frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2y\cdot\frac{dy}{dx}=\frac{1}{\sqrt{x^2+y^2}}\left(x+y\frac{dy}{dx}\right) \qquad (1)$$

From the relation $x^3+y^3+3axy=5a^3$, we have

$$\frac{dy}{dx}=\frac{-\frac{\partial}{\partial x}(x^3+y^3+3axy-5a^3)}{\frac{\partial}{\partial y}(x^3+y^3+3axy-5a^3)}=\frac{-(3x^2+3ay)}{3y^2+3ax}=-1 \text{ at } (a,\, a)$$

Putting $\frac{dy}{dx}=-1$ in Eq. (1), we get

$$\left(\frac{dz}{dx}\right)_{(a,a)}=\left[\frac{1}{\sqrt{x^2+y^2}}\{x+y(-1)\}\right]_{(a,a)}=0$$

Example 2.14 If $w=f(x,y), x=r\cos\theta, y=r\sin\theta$, show that

$$\left(\frac{\partial w}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2=\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2$$

Solution: The given equation defines w as a composite function of r and θ.

$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r}=\frac{\partial w}{\partial x}\cdot\cos\theta+\frac{\partial w}{\partial y}\cdot\sin\theta$$

or $$\frac{\partial w}{\partial r}=\frac{\partial f}{\partial x}\cos\theta+\frac{\partial f}{\partial y}\sin\theta \qquad [\because \quad w=f(x,y)] \qquad (1)$$

Also, $$\frac{\partial w}{\partial \theta}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta}=\frac{\partial w}{\partial x}(-r\sin\theta)+\frac{\partial w}{\partial y}(r\cos\theta)$$

or $$\frac{1}{r}\frac{\partial w}{\partial \theta}=-\frac{\partial f}{\partial x}\sin\theta+\frac{\partial f}{\partial y}\cos\theta \qquad (2)$$

Squaring and adding (1) and (2), we get

$$\left(\frac{\partial w}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2=\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2$$

Example 2.15 If $u=u\left(\frac{y-x}{xy},\frac{z-x}{xz}\right)$, show that $x^2\frac{\partial u}{\partial x}+y^2\frac{\partial u}{\partial y}+z^2\frac{\partial u}{\partial z}=0$.

Solution: Let $v=\frac{y-x}{xy}=\frac{1}{x}-\frac{1}{y}$ and $w=\frac{z-x}{xz}=\frac{1}{x}-\frac{1}{z}$ (1)

so that $u=u(v,w)$. Therefore,

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial x}=\frac{\partial u}{\partial v}\left(-\frac{1}{x^2}\right)+\frac{\partial u}{\partial w}\left(-\frac{1}{x^2}\right) \qquad \text{[using (1)]}$$

or $$x^2\frac{\partial u}{\partial x}=-\frac{\partial u}{\partial v}-\frac{\partial u}{\partial w} \quad (2)$$

Also, $$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial y}+\frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial y}=\frac{\partial u}{\partial v}\left(\frac{1}{y^2}\right)+\frac{\partial u}{\partial w}(0) \quad \text{[using (1)]}$$

or $$y^2\frac{\partial u}{\partial y}=\frac{\partial u}{\partial v} \quad (3)$$

Similarly, $$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial z}+\frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial z}=\frac{\partial u}{\partial v}(0)+\frac{\partial u}{\partial w}\left(\frac{1}{z^2}\right) \quad \text{[using (1)]}$$

or $$z^2\frac{\partial u}{\partial z}=\frac{\partial u}{\partial w} \quad (4)$$

Adding (2), (3) and (4), we have

$$x^2\frac{\partial u}{\partial x}+y^2\frac{\partial u}{\partial y}+z^2\frac{\partial u}{\partial z}=0$$

Example 2.16 If $u=(x-y)f\left(\frac{y}{x}\right)$ find $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$

Solution: Given : $$\frac{\partial u}{\partial x}=(x-y)f'\left(\frac{y}{x}\right)\left[\frac{-y}{x^2}\right]+f\left(\frac{y}{x}\right)$$

$$=-\left[\frac{y}{x^2}\right](x-y)f'\left(\frac{y}{x}\right)+f\left(\frac{y}{x}\right)$$

$$\frac{\partial^2 u}{\partial x^2}=\left(-\frac{y}{x^2}\right)(x-y)f''\left(\frac{y}{x}\right)\left[\frac{-y}{x^2}\right]+\left(\frac{-y}{x^2}\right)f'\left(\frac{y}{x}\right)+\left[\frac{2y}{x^3}\right](x-y)f'\left(\frac{y}{x}\right)+f'\left(y/x\right)\left[\frac{-y}{x^2}\right]$$

$$x^2\frac{\partial^2 u}{\partial x^2}=\left(-\frac{y^2}{x}\right)(x-y)f''\left(\frac{y}{x}\right)-yf'\left(\frac{y}{x}\right)+\left[\frac{2y}{x}\right](x-y)f'\left(\frac{y}{x}\right)-yf'\left(y/x\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2}=\left(\frac{y}{x^2}\right)(x-y)f''\left(\frac{y}{x}\right)-2yf'\left(\frac{y}{x}\right)+\left[\frac{2y}{x}\right](x-y)f'\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial y}=(x-y)f'\left(\frac{y}{x}\right)\left[\frac{1}{x}\right]+f\left(\frac{y}{x}\right)(-1)$$

$$=\frac{1}{x}(x-y)f'\left(\frac{y}{x}\right)-f\left(\frac{y}{x}\right)$$

$$\frac{\partial^2 u}{\partial y^2}=\left(\frac{1}{x}\right)(x-y)f''\left(\frac{y}{x}\right)\left[\frac{1}{x}\right]+\left(\frac{1}{x}\right)f'\left(\frac{y}{x}\right)(-1)-f'\left(\frac{y}{x}\right)\frac{1}{x}$$

$$=\left(\frac{1}{x^2}\right)(x-y)f''\left(\frac{y}{x}\right)-\left(\frac{2}{x}\right)f'\left(\frac{y}{x}\right)$$

$$y^2\frac{\partial^2 u}{\partial y^2}=\left(\frac{y^2}{x^2}\right)(x-y)f''\left(\frac{y}{x}\right)-\frac{2y^2}{x}f'\left(\frac{y}{x}\right)$$

$$\frac{\partial^2 u}{\partial x\partial y}=\left(\frac{1}{x}\right)(x-y)f''\left(\frac{y}{x}\right)\left[\frac{-y}{x^2}\right]+\left(\frac{1}{x}\right)f'\left(\frac{y}{x}\right)+\left[\frac{-1}{x^2}\right](x-y)f'\left(\frac{y}{x}\right)-f'\left(\frac{y}{x}\right)\left[\frac{-y}{x^2}\right]$$

$$=\left(\frac{-y}{x^3}\right)(x-y)f'\left(\frac{y}{x}\right)+\left(\frac{1}{x}\right)f'\left(\frac{y}{x}\right)+\left[\frac{-1}{x^2}\right](x-y)f'\left(\frac{y}{x}\right)-f'\left(\frac{y}{x}\right)\left[\frac{-y}{x^2}\right]$$

$$2xy\frac{\partial^2 u}{\partial x\partial y}=\left(\frac{-2y^2}{x^3}\right)(x-y)f'\left(\frac{y}{x}\right)+2yf'\left(\frac{y}{x}\right)-\left[\frac{2y(x-y)}{x}\right]f'\left(\frac{y}{x}\right)+\frac{2y^2}{x}f'\left(\frac{y}{x}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0$$

Example 2.17 If $W=f(y-z, z-x, x-y)$ show that $\frac{\partial W}{\partial x}+\frac{\partial W}{\partial y}+\frac{\partial W}{\partial z}=0$

Solution: Let $u=y-z$, $v=z-x$, $w=x-y$, $W=f(u, v, w)$

$$\frac{\partial W}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial f}{\partial w}\frac{\partial w}{\partial x}=\frac{\partial f}{\partial u}(0)+\frac{\partial f}{\partial v}(-1)+\frac{\partial f}{\partial w}(1)=-\frac{\partial f}{\partial v}+\frac{\partial f}{\partial w}$$

$$\frac{\partial W}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial f}{\partial w}\frac{\partial w}{\partial y}=\frac{\partial f}{\partial u}(1)+\frac{\partial f}{\partial v}(0)+\frac{\partial f}{\partial w}(-1)=\frac{\partial f}{\partial u}-\frac{\partial f}{\partial w}$$

$$\frac{\partial W}{\partial z}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial z}\frac{\partial f}{\partial v}\frac{\partial v}{\partial z}+\frac{\partial f}{\partial w}\frac{\partial w}{\partial z}=\frac{\partial f}{\partial u}(-1)+\frac{\partial f}{\partial v}(1)+\frac{\partial f}{\partial w}(0)=-\frac{\partial f}{\partial u}+\frac{\partial f}{\partial v}$$

Adding we get $\frac{\partial W}{\partial x}+\frac{\partial W}{\partial y}+\frac{\partial W}{\partial z}=0$

2.4 CHANGE OF VARIABLES

Let $z=f(x,y)$ (2.20)

where $x=\phi(u,v)$ and $y=\psi(u,v)$ (2.21)

More often, it is required to change expressions containing z, x, y, $\partial z/\partial x$, $\partial z/\partial y$, etc., to expressions containing z, u, v, $\partial z/\partial u$, $\partial z/\partial v$, etc. Then we can obtain the necessary formulae for the change of variables. The variables x, y, z will be functions of u alone when v is treated as a constant. Thus, by chain rule, we have

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \tag{2.22}$$

where the ordinary derivatives are being changed to the partial derivatives since x, y are functions of two variables u and v. Similarly, treating u as a constant, we obtain $\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$ (2.23)

Solving (2.22) and (2.23) for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, we get the values in terms of $\frac{\partial z}{\partial u}$, $\frac{\partial z}{\partial v}$, z, u, v.

Note:

(i) If instead of Eq. (2.21), we are given that $u = \phi(x, y)$ and $v = \psi(x, y)$, then (2.22) and (2.23) become

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

(ii) By repeatedly applying formulae (2.22) and (2.23) or the formulas in Note (i), we can obtain the higher derivatives of z.

Example 2.18 If $u = F(x-y, y-z, z-x)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Solution: Put $x - y = r, y - z = s,$ and $z - x = t$, so that $u = f(r, s, t)$. Therefore,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r} \cdot (1) + \frac{\partial x}{\partial s} \cdot (0) + \frac{\partial u}{\partial t} \cdot (-1) = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} \qquad (1)$$

Similarly, $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial y} = -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} \qquad (2)$$

and $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial z} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial z} = -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} \qquad (3)$$

Adding (1), (2) and (3), we get $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Example 2.19 If $z = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$ show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

Solution: Here, $$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta \qquad (1)$$

and $$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial \theta} = \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)$$

$\Rightarrow$ $$\frac{1}{r}\frac{\partial z}{\partial \theta} = -\sin\theta\frac{\partial z}{\partial x} + \cos\theta\frac{\partial z}{\partial y} \qquad (2)$$

Squaring and adding (1) and (2), we get

$$\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2=\left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta+\left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta+2\sin\theta\cos\theta\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$

$$+\left(\frac{\partial z}{\partial x}\right)^2\sin^2\theta+\left(\frac{\partial z}{\partial y}\right)^2\cos^2\theta-2\sin\theta\cos\theta\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$

$$=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2$$

Example 2.20 Transform the equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ into polar coordinates.

Solution: The relation which connects Cartesian coordinates (x, y) with polar coordinates (r, θ) are $x=r\cos\theta$, $y=r\sin\theta$ so that $r^2=x^2+y^2$ and $\tan\theta=\frac{y}{x}$. Thus,

$$r=\sqrt{x^2+y^2} \text{ and } \theta=\tan^{-1}\left(\frac{y}{x}\right)$$

$$\therefore \quad \frac{\partial r}{\partial x}=\frac{x}{\sqrt{x^2+y^2}}=\frac{r\cos\theta}{r}=\cos\theta,\ \frac{\partial r}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}=\frac{r\sin\theta}{r}=\sin\theta$$

and
$$\frac{\partial \theta}{\partial x}=\frac{1}{1+\frac{y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right)=-\frac{y}{x^2+y^2}=-\frac{r\sin\theta}{r^2}=-\frac{\sin\theta}{r};\ \frac{\partial \theta}{\partial y}=\frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x}=\frac{x}{x^2+y^2}=\frac{r\cos\theta}{r^2}=\frac{\cos\theta}{r}$$

Here, u is a composite function of x and y.

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}=\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\cdot\frac{\partial u}{\partial \theta}$$

$$\frac{\partial}{\partial x}(u)=\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\cdot\frac{\partial}{\partial \theta}\right)u$$

or
$$\frac{\partial}{\partial x}\equiv\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\cdot\frac{\partial}{\partial \theta} \qquad (1)$$

Also,
$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}=\sin\theta\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\cdot\frac{\partial u}{\partial \theta}$$

or
$$\frac{\partial}{\partial y}(u)=\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\cdot\frac{\partial}{\partial \theta}\right)u \quad \text{or} \quad \frac{\partial}{\partial y}\equiv\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\cdot\frac{\partial}{\partial \theta} \qquad (2)$$

Now, we will make use of the equivalence of Cartesian and polar operators as given by (1) and (2).

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}.\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\cdot\frac{\partial u}{\partial\theta}\right)$$

$$=\cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right)-\frac{\sin\theta}{r}\cdot\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\cdot\frac{\partial u}{\partial\theta}\right)$$

$$=\cos\theta\left[\cos\theta\frac{\partial^2 u}{\partial r^2}-\sin\theta\frac{\partial u}{\partial\theta}\left(-\frac{1}{r^2}\right)-\frac{\sin\theta}{r}\cdot\frac{\partial^2 u}{\partial r\partial\theta}\right]$$

$$-\frac{\sin\theta}{r}\left[-\sin\theta\frac{\partial u}{\partial r}+\cos\theta\frac{\partial^2 u}{\partial\theta\partial r}-\frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}-\frac{\sin\theta}{r}\cdot\frac{\partial^2 u}{\partial\theta^2}\right]$$

$$=\cos^2\theta\frac{\partial^2 u}{\partial r^2}+\frac{2\cos\theta\sin\theta}{r^2}\cdot\frac{\partial u}{\partial\theta}+\frac{\sin^2\theta}{r}\cdot\frac{\partial u}{\partial r}-\frac{2\cos\theta\sin\theta}{r}\cdot\frac{\partial^2 u}{\partial r\partial\theta}+\frac{\sin^2\theta}{r^2}\cdot\frac{\partial^2 u}{\partial\theta^2} \quad (3)$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\cdot\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\cdot\frac{\partial u}{\partial\theta}\right)$$

$$=\sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\cdot\frac{\partial u}{\partial\theta}\right)+\frac{\cos\theta}{r}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\cdot\frac{\partial u}{\partial\theta}\right)$$

$$=\sin\theta\left[\sin\theta\frac{\partial^2 u}{\partial r^2}-\frac{\cos\theta}{r^2}\cdot\frac{\partial u}{\partial\theta}+\frac{\cos\theta}{r}\cdot\frac{\partial^2 u}{\partial r\cdot\partial u}\right]+\frac{\cos\theta}{r}\left[\cos\theta\frac{\partial u}{\partial r}+\sin\theta\frac{\partial^2 u}{\partial r\cdot\partial u}-\frac{\sin\theta}{r}\cdot\frac{\partial u}{\partial\theta}+\frac{\cos\theta}{r}\cdot\frac{\partial^2 u}{\partial\theta^2}\right]$$

$$=\sin^2\theta\frac{\partial^2 u}{\partial r^2}-\frac{2\cos\theta\sin\theta}{r^2}\cdot\frac{\partial u}{\partial\theta}+\frac{\cos^2\theta}{r}\cdot\frac{\partial u}{\partial r}+\frac{2\cos\theta\sin\theta}{r}\cdot\frac{\partial^2 u}{\partial r\partial\theta}+\frac{\cos^2\theta}{r^2}\cdot\frac{\partial^2 u}{\partial\theta^2} \quad (4)$$

Adding (3) and (4), we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial\theta^2}$$

Therefore, $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ transforms into $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial u}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial\theta^2}=0$.

Example 2.21 If $x+y=2e^{\theta}\cos\phi$ and $x-y=2ie^{\theta}\sin\phi$, show that $\frac{\partial^2 u}{\partial\theta^2}+\frac{\partial^2 u}{\partial\phi^2}=4xy\frac{\partial^2 u}{\partial x\partial y}$.

Solution: We have $x=e^{\theta}(\cos\phi+i\sin\phi)=e^{\theta}\cdot e^{i\phi}$

and $y=e^{\theta}(\cos\phi-i\sin\phi)=e^{\theta}\cdot e^{-i\phi}$

Here, u is a composite function of θ and ϕ. Therefore,

$$\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$=\frac{\partial u}{\partial x}(e^{\theta}\cdot e^{i\phi})+\frac{\partial u}{\partial y}(e^{\theta}\cdot e^{-i\phi})=x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$$

or $$\frac{\partial}{\partial \theta}=x\frac{\partial}{\partial u}+y\frac{\partial}{\partial y} \qquad (1)$$

Also, $$\frac{\partial u}{\partial \phi}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \phi}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \phi}$$

$$=\frac{\partial u}{\partial x}\cdot(ie^{\theta}\cdot e^{i\phi})+\frac{\partial u}{\partial y}\cdot(e^{\theta}\cdot -ie^{-i\phi})=ix\frac{\partial u}{\partial x}-iy\frac{\partial u}{\partial y}$$

or $$\frac{\partial}{\partial \phi}=ix\frac{\partial}{\partial x}-iy\frac{\partial}{\partial y} \qquad (2)$$

Using the operator (1), we have

$$\frac{\partial^2 u}{\partial \theta^2}=\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right)=\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$=x\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)+x\frac{\partial}{\partial x}\left(y\frac{\partial u}{\partial y}\right)+y\frac{\partial}{\partial y}\left(x\frac{\partial u}{\partial x}\right)+y\frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial y}\right)$$

$$=x\left(x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\right)+xy\frac{\partial^2 u}{\partial x\partial y}+yx\frac{\partial^2 u}{\partial y\partial x}+y\left(y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}\right)$$

$$=x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} \qquad (3)$$

Similarly, using operator (2), we get

$$\frac{\partial^2 u}{\partial \phi^2}=\frac{\partial}{\partial \phi}\left(\frac{\partial u}{\partial \phi}\right)=\left(ix\frac{\partial}{\partial x}-iy\frac{\partial}{\partial y}\right)\left(ix\frac{\partial u}{\partial x}-iy\frac{\partial u}{\partial y}\right)$$

$$=-x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}-y^2\frac{\partial^2 u}{\partial y^2}-x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y} \qquad (4)$$

Adding (3) and (4), we get

$$\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial \phi^2}=4xy\frac{\partial^2 u}{\partial x\partial y}$$

2.5 JACOBIANS

Let u and v be functions of two independent variables x and y. Then the determinant $\begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$ is called the **Jacobian** of u and v with respect to x and y. It is denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ or $J\left(\frac{u,v}{x,y}\right)$.

Also, the Jacobian of u, v and w with respect to x, y and z is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} \text{ or } J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix}$$

Similarly, we can define the Jacobian of more than three variables. The term 'Jacobian' was named after German mathematician Carl Gustav Jacob Jacobi (1804–1851) who made significant contribution to mechanics, partial differential equations, astronomy, elliptic functions and the calculus of variations.

2.5.1 Chain Rule for Jacobians

If u, v are functions of x, y and x, y are functions of r, s, then

$$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,s)} = \frac{\partial(u,v)}{\partial(r,s)}$$

Let us prove this result.

$$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,s)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} \begin{vmatrix} \partial x/\partial r & \partial x/\partial s \\ \partial y/\partial r & \partial y/\partial s \end{vmatrix}$$

$$= \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} \begin{vmatrix} \partial x/\partial r & \partial y/\partial r \\ \partial x/\partial s & \partial y/\partial s \end{vmatrix} \quad \text{(Interchanging rows and columns)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} & \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} \\ \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial r} & \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial s} \end{vmatrix}$$

Since $u = f(x,y)$, $v = g(x,y)$, where $x = \phi(r,s)$, $y = \psi(r,s)$, thus by chain rule, we have

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}, \quad \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial r}, \quad \frac{\partial v}{\partial s} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial s}$$

Hence, $$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,s)} = \begin{vmatrix} \partial u/\partial r & \partial u/\partial s \\ \partial v/\partial r & \partial v/\partial s \end{vmatrix} = \frac{\partial(u,v)}{\partial(r,s)}$$

In general, $\dfrac{\partial(x_1,x_2,\ldots,x_n)}{\partial(u_1,u_2,\ldots,u_n)}\cdot\dfrac{\partial(u_1,u_2,\ldots,u_n)}{\partial(v_1,v_2,\ldots,v_n)}=\dfrac{\partial(x_1,x_2,\ldots,x_n)}{\partial(v_1,v_2,\ldots,v_n)}$

Corollary: If $J=\dfrac{\partial(u,v)}{\partial(x,y)}$ and $J'=\dfrac{\partial(x,y)}{\partial(u,v)}$, then $JJ'=1$.

Proof: If we replace r, s by u, v in the previous result, then we get the required result. In general,

$$\frac{\partial(u_1,u_2,\ldots,u_n)}{\partial(x_1,x_2,\ldots,x_n)}\cdot\frac{\partial(x_1,x_2,\ldots,x_n)}{\partial(u_1,u_2,\ldots,u_n)}=1$$

Example 2.22 If $x=r\cos\theta$, $y=r\sin\theta$, evaluate $\dfrac{\partial(x,y)}{\partial(r,\theta)}$ and $\dfrac{\partial(r,\theta)}{\partial(x,y)}$.

Solution: Since $x=r\cos\theta$, $y=r\sin\theta$, we have

$$\frac{\partial x}{\partial r}=\cos\theta,\ \frac{\partial y}{\partial r}=\sin\theta \text{ and } \frac{\partial x}{\partial\theta}=-r\sin\theta,\ \frac{\partial y}{\partial\theta}=r\cos\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial\theta}\\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial\theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}$$

$$=r\cos^2\theta+r\sin^2\theta=r(\cos^2\theta+\sin^2\theta)=r$$

Now, $r^2=x^2+y^2$ and $\theta=\tan^{-1}\dfrac{y}{x}$

$$\frac{\partial r}{\partial x}=\frac{x}{r},\ \frac{\partial r}{\partial y}=\frac{y}{r} \text{ and } \frac{\partial\theta}{\partial x}=\frac{-y}{x^2+y^2}=-\frac{y}{r^2},\ \frac{\partial\theta}{\partial y}=\frac{x}{x^2+y^2}=\frac{x}{r^2}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)}=\begin{vmatrix}\dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y}\\ \dfrac{\partial\theta}{\partial x} & \dfrac{\partial\theta}{\partial y}\end{vmatrix}=\begin{vmatrix}\dfrac{x}{r} & \dfrac{y}{r}\\ \dfrac{-y}{r^2} & \dfrac{x}{r^2}\end{vmatrix}=\frac{x^2}{r^3}+\frac{y^2}{r^3}=\frac{x^2+y^2}{r^3}=\frac{r^2}{r^3}=\frac{1}{r}$$

Note: $\dfrac{\partial(x,y)}{\partial(r,\theta)}\times\dfrac{\partial(r,\theta)}{\partial(x,y)}=r\cdot\dfrac{1}{r}=1$

Example 2.23 If $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$ and $z=r\cos\theta$, show that $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=r^2\sin\theta$.

Solution: We have $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=\begin{vmatrix}\partial x/\partial r & \partial x/\partial\theta & \partial x/\partial\phi\\ \partial y/\partial r & \partial y/\partial\theta & \partial y/\partial\phi\\ \partial z/\partial r & \partial z/\partial\theta & \partial z/\partial\phi\end{vmatrix}=\begin{vmatrix}\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi\\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi\\ \cos\theta & -r\sin\theta & 0\end{vmatrix}$

Taking out common factors (r from the second column and $r\sin\theta$ from the third column), we get

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta\begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

Expanding by third row, we get

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta\left\{\cos\theta\begin{vmatrix} \cos\theta\cos\phi & -\sin\phi \\ \cos\theta\sin\phi & \cos\phi \end{vmatrix} + \sin\theta\begin{vmatrix} \sin\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\phi \end{vmatrix}\right\}$$

$$= r^2\sin\theta\,[\cos\theta\,(\cos\theta\cos^2\phi + \cos\theta\sin^2\phi) + \sin\theta\,(\sin\theta\cos^2\phi + \sin\theta\sin^2\phi)]$$

$$= r^2\sin\theta(\cos^2\theta + \sin^2\theta) = r^2\sin\theta$$

Note: Here (x,y,z) and (r,θ,ϕ) are the Cartesian and spherical polar coordinates of a point, respectively.

Example 2.24 If $y_1 = \frac{x_2x_3}{x_1}, y_2 = \frac{x_3x_1}{x_2}, y_3 = \frac{x_1x_2}{x_3}$, show that the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3 is 4.

Solution: The Jacobian of y_1, y_2, y_3 w.r.t. x_1, x_2, x_3 is given by

$$\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} \frac{-x_2x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & \frac{-x_3x_1}{x_2^2} & \frac{x_1}{x_1} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & \frac{-x_1x_2}{x_3^2} \end{vmatrix}$$

Taking out common $\frac{1}{x_1^2}, \frac{1}{x_2^2}$ and $\frac{1}{x_3^2}$ from first, second and third row, respectively, we get

$$\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = \frac{1}{x_1^2x_2^2x_3^2}\begin{vmatrix} -x_2x_3 & x_3x_1 & x_1x_2 \\ x_2x_3 & -x_3x_1 & x_1x_2 \\ x_2x_3 & x_3x_1 & -x_1x_2 \end{vmatrix}$$

Taking out common x_2x_3, x_3x_1 and x_1x_2 from the first, second and third column, respectively, we get

$$\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = \frac{x_1^2x_2^2x_3^3}{x_1^2x_2^2x_3^3}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -1(1-1) - 1(-1-1) + 1(1+1) = 0+2+2 = 4$$

Example 2.25 If $u = x^2 - y^2$, $v = 2xy$ and $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial(u,v)}{\partial(r,\theta)}$.

Solution: We have $\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$

Since $u = x^2 - y^2, v = 2xy$, we have

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2 + y^2) \quad (1)$$

Since $x = r\cos\theta, y = r\sin\theta$, we have

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \quad (2)$$

Hence, $\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)} = 4(x^2+y^2)\cdot r = 4(r^2\cos^2\theta + r^2\sin^2\theta)\cdot r = 4r^3$ [using (1) and (2)]

2.5.2 Jacobians of Implicit Functions

If u_1 and u_2 are implicit functions of the variables x_1 and x_2 connected by the relations $f_1(u_1,u_2,x_1,x_2) = 0$ and $f_2(u_1,u_2,x_1,x_2) = 0$, then $\frac{\partial(u_1,u_2)}{\partial(x_1,x_2)} = (-1)^2 \frac{\partial(f_1,f_2)/\partial(x_1,x_2)}{\partial(f_1,f_2)/\partial(u_1,u_2)}$. In general,

$$\frac{\partial(u_1,u_2,\ldots,u_n)}{\partial(x_1,x_2,\ldots,x_n)} = (-1)^n \frac{\partial(f_1,f_2,\ldots,f_n)/\partial(x_1,x_2,\ldots,x_n)}{\partial(f_1,f_2,\ldots,f_n)/\partial(u_1,u_2,\ldots,u_n)}$$

Note: This result bears resemblance to the result $\frac{\partial y}{\partial x} = \frac{-\partial f/\partial x}{\partial f/\partial y}$, where x and y are connected by the relation $f(x,y) = 0$.

Example 2.26 If $u = xyz,\ v = x^2 + y^2 + z^2$ and $w = x + y + z$, find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

Solution: Let us first calculate the value of J.

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} yz & zx & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} yz & z(x-y) & y(x-z) \\ 2x & 2(y-x) & 2(z-x) \\ 1 & 0 & 0 \end{vmatrix}$$

$$= 2z(x-y)(z-x) - 2y(y-x)(x-z) = 2(x-y)(x-z)(y-z)$$

Since $JJ' = 1$, we have

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = J' = \frac{1}{J} = \frac{1}{2(x-y)(y-z)(x-z)}$$

Example 2.27 If $u = \frac{x}{y-z}, v = \frac{y}{z-x}, w = \frac{z}{x-y}$, show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$.

Solution: We have $u = \frac{x}{y-z}, v = \frac{y}{z-x}, w = \frac{z}{x-y}$. Therefore,

$$\log u = \log x - \log(y-z) \quad (1)$$

$$\log v = \log y - \log(z-x) \quad (2)$$

$$\log w = \log z - \log(x-y) \quad (3)$$

Differentiating (1) partially w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{\partial u}{\partial x} = \frac{1}{x} \quad \text{or} \quad \frac{\partial u}{\partial x} = \frac{u}{x}$$

Differentiating (1) partially w.r.t. y, we get

$$\frac{1}{u}\cdot\frac{\partial u}{\partial y} = -\frac{1}{y-z} \quad \text{or} \quad \frac{\partial u}{\partial y} = \frac{-u}{y-z}$$

Differentiating (1) partially w.r.t. z, we get

$$\frac{1}{u}\cdot\frac{\partial u}{\partial z} = \frac{-1}{y-z}(-1) \quad \text{or} \quad \frac{\partial u}{\partial z} = \frac{u}{y-z}$$

Similarly, from (2) and (3), we have

$$\frac{\partial v}{\partial x} = \frac{v}{z-x}, \frac{\partial v}{\partial y} = \frac{v}{y}, \frac{\partial v}{\partial z} = \frac{-v}{z-x}, \frac{\partial w}{\partial x} = \frac{-w}{x-y}, \frac{\partial w}{\partial y} = \frac{w}{x-y}, \frac{\partial w}{\partial z} = \frac{w}{z}$$

$$\therefore \quad \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{u}{x} & \frac{-u}{y-z} & \frac{u}{y-z} \\ \frac{v}{z-x} & \frac{v}{y} & \frac{-v}{z-x} \\ \frac{-w}{x-y} & \frac{w}{x-y} & \frac{w}{z} \end{vmatrix}$$

Taking out common u, v, w from R_1, R_2, R_3 respectively, we get

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = uvw\begin{vmatrix} \frac{1}{x} & \frac{-1}{y-z} & \frac{1}{y-z} \\ \frac{1}{z-x} & \frac{1}{y} & \frac{-1}{z-x} \\ \frac{-1}{x-y} & \frac{1}{x-y} & \frac{1}{z} \end{vmatrix}$$

Multiplying R_1, R_2, R_3 by $y-z$, $z-x$, $x-y$, respectively, we get

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{uvw}{(y-z)(z-x)(x-y)}\begin{vmatrix} \frac{y-z}{x} & -1 & 1 \\ 1 & \frac{z-x}{y} & -1 \\ -1 & 1 & \frac{x-y}{z} \end{vmatrix}$$

Multiplying C_1, C_2, C_3 by x, y, z, respectively, we get

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{uvw}{xyz(y-z)(z-x)(x-y)}\begin{vmatrix} y-z & -y & z \\ x & z-x & -z \\ -x & y & x-y \end{vmatrix}$$

Operating $C_1 \to C_1 + C_2 + C_3$, we get

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1}{(y-z)^2(z-x)^2(x-y)^2}\begin{vmatrix} 0 & -y & z \\ 0 & z-x & -z \\ 0 & y & x-y \end{vmatrix} = 0$$

2.5.3 Functional Relationship

Let u_1, u_2, u_3 be the functions of x_1, x_2, x_3, respectively. Then the functional relationship of the form $f(u_1, u_2, u_3) = 0$ exists if $J\left(\frac{u_1, u_2, u_3}{x_1, x_2, x_3}\right) = 0$ and conversely, if the functional relationship of the form $f(u_1, u_2, u_3) = 0$ exists, then $J\left(\frac{u_1, u_2, u_3}{x_1, x_2, x_3}\right) = 0$.

Example 2.28 Show that the functions $u = x + y + z$, $v = x^3 + y^3 + z^3 - 3xyz$ and $w = x^2 + y^2 + z^2 - xy - yz - zx$ are functionally dependent and find the relation between them.

Solution: The functions u, v, w are functionally dependent if $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$. Therefore,

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 3(x^2-yz) & 3(y^2-xz) & 3(z^2-xy) \\ 2x-y-z & 2y-z-x & 2z-x-y \end{vmatrix}$$

$$= 3\begin{vmatrix} 1 & 0 & 0 \\ x^2-yz & y^2-x^2+z(y-x) & z^2-x^2+y(z-x) \\ 2x-y-z & 3(y-x) & 3(z-x) \end{vmatrix}$$

Expanding along R_1, we get

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 9\begin{vmatrix} (y-x)(x+y+z) & (z-x)(x+y+z) \\ y-x & z-x \end{vmatrix} = 9(y-x)(z-x)(x+y+z)\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

Therefore, u, v, w are functionally dependent.

Now, $v = x^3+y^3+z^3-3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx) = uw$.

Hence, $v = uw$ is the desired relation.

Example 2.29 If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$ show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$

Solution:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix} = \left(\frac{1}{x^2}\right)\left(\frac{1}{y^2}\right)\left(\frac{1}{z^2}\right)\begin{vmatrix} -yz & xz & xy \\ zy & -zx & xy \\ yz & xz & -xy \end{vmatrix}$$

$$= \frac{-yz}{x^2}\left(\frac{x^2yz}{y^2z^2} - \frac{x^2}{yz}\right) - \frac{z}{x}\left(\frac{-xy}{yz} - \frac{x}{z}\right) + \frac{y}{x}\left(\frac{x}{y} + \frac{x}{y}\right) = 0+2+2 = 4$$

Example 2.30 Prove $u = x + y + z$; $v = xy + yz + zx$; $w = x^2 + y^2 + z^2$ are functionally independent. Find the relationship between them.

Solution:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ y+z & z+x & x+y \\ 2x & 2y & 2z \end{vmatrix}$$

$$= 1\left[2z(z+x) - 2y(x+y)\right] - 1\left[2z(y+z) - 2x(x+y)\right] + 1\left[2y(y+z) - 2x(z+x)\right]$$

$$= 2z^2 + 2xz - 2xy - 2y^2 - 2yz - 2z^2 + 2x^2 + 2xy + 2y^2 + 2yz - 2xz - 2x^2 = 0$$

u,v and w are functionally dependent.
The relation between them is given is $(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$

$$u^2 = w + 2v$$

2.6 TAYLOR'S SERIES FOR FUNCTIONS OF TWO VARIABLES

Theorem 2.4 *If $f(x, y)$ possesses finite and continuous partial derivatives of all orders, then*

$$f(x+h, y+k) = f(x,y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \cdots$$

Proof: We know that Taylor's theorem for a function $f(x)$ of single variable x is

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots$$

Now, expanding $f(x+h, y+k)$ as a single variable x by keeping y as constant, we have

$$f(x+h, y+k) = f(x, y+k) + h\frac{\partial}{\partial x}f(x, y+k) + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}f(x, y+k) + \cdots \quad (2.24)$$

and expanding $f(x+h, y+k)$ as a single variable y by keeping x as constant, we get

$$f(x, y+k) = f(x,y) + k\frac{\partial}{\partial y}f(x,y) + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2}f(x,y) + \cdots \quad (2.25)$$

Using (2.25), (2.24) can be written as

$$f(x+h, y+k)$$
$$=\left[f(x,y)+k\frac{\partial}{\partial y}f(x,y)+\frac{k^2}{2!}\frac{\partial^2}{\partial y^2}f(x,y)+\cdots\right]+h\frac{\partial}{\partial x}\left[f(x,y)+k\frac{\partial}{\partial y}f(x,y)+\frac{k^2}{2!}\frac{\partial^2}{\partial y^2}f(x,y)+\cdots\right]$$
$$+\frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left[f(x,y)+k\frac{\partial}{\partial y}f(x,y)+\frac{k^2}{2!}\frac{\partial^2}{\partial y^2}f(x,y)+\cdots\right]+\cdots$$

Hence, $$f(x+h,y+k)=f(x,y)+\left(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right)+\frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\partial y}+k^2\frac{\partial^2 f}{\partial y^2}\right)+\cdots \tag{2.26}$$

Symbolically, this result can be written as

$$f(x+h,y+k)=f(x,y)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f+\cdots+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f+\cdots$$

Corollary:

(i) Putting $x = a$ and $y = b$, (2.26) becomes

$$f(a+h,b+k)=f(a,b)+[hf_x(a,b)+kf_y(a,b)]+\frac{1}{2!}[h^2 f_{xx}(a,b)+2hkf_{xy}(a,b)+k^2 f_{yy}(a,b)]+\cdots$$

Putting $a + h = x$ and $b + k = y$ so that $h = x – a$, $k = y – b$, we get

$$f(x,y)=f(a,b)+[(x-a)f_x(a,b)+(y-b)f_y(a,b)]$$
$$+\frac{1}{2!}[(x-a)^2 f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-a)^2 f_{yy}(a,b)]+\cdots \tag{2.27}$$

which is a Taylor's expansion of $f(x, y)$ in powers of $(x – a)$ and $(y – b)$. This is used for the expansion of $f(x, y)$ in the neighbourhood of (a, b).

(ii) Putting $a = 0$, $b = 0$ in (2.27), we get

$$f(x,y)=f(0,0)+[xf_x(0,0)+yf_y(0,0)]+\frac{1}{2!}[x^2 f_{xx}(0,0)+2xyf_{xy}(0,0)+y^2 f_{yy}(0,0)]+\cdots \tag{2.28}$$

This is a **Maclaurin's expansion** of $f(x, y)$. This is used for the expansion of $f(x, y)$ in the neighbourhood of origin (0, 0).

Example 2.31 Expand $e^x \sin y$ in powers of x and y as far as terms of the third degree.

Solution: Let $f(x,y)=e^x \sin y$ or $f(0,0)=0$

$f_x(x,y)=e^x \sin y$ or $f_x(0,0)=0$; $f_y(x,y)=e^x \cos y$ or $f_y(0,0)=1$

$f_{xx}(x,y)=e^x \sin y$ or $f_{xx}(0,0)=0$; $f_{xy}(x,y)=e^x \cos y$ or $f_{xy}(0,0)=1$

$f_{yy}(x,y)=-e^x \sin y$ or $f_{yy}(0,0)=0$; $f_{xxx}(x,y)=e^x \sin y$ or $f_{xxx}(0,0)=0$

$f_{xxy}(x,y)=e^x \cos y$ or $f_{xxy}(0,0)=1$; $f_{xyy}(x,y)=-e^x \sin y$ or $f_{xyy}(0,0)=0$

$$f_{yyy}(x,y) = -e^x \cos y \quad \text{or} \quad f_{yyy}(0,0) = -1$$

Maclaurin's expansion of $f(x,y)$ is given by

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)]$$

$$+\frac{1}{3!}[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)] + \cdots$$

$$= 0 + (x\cdot 0 + y\cdot 1) + \frac{1}{2!}(x^2\cdot 0 + 2xy\cdot 1 + y^2\cdot 0) + \frac{1}{3!}[x^3\cdot 0 + 3x^2 y\cdot 1 + 3xy^2\cdot 0 + y^3(-1)] + \cdots$$

$$= y + xy + \frac{1}{2}x^2 y - \frac{1}{6}y^3 + \cdots$$

Example 2.32 Expand $x^2 y + 3y - 2$ in powers of $(x - 1)$ and $(y + 2)$ using Taylor's theorem.

Solution: Let $f(x,y) = x^2 y + 3y - 2 \Rightarrow f(1,-2) = -10$. Therefore,

$$f_x(x,y) = 2xy \quad \text{or} \quad f_x(1,-2) = -4; \; f_y(x,y) = x^2 + 3 \quad \text{or} \quad f_y(1,-2) = 4;$$

$$f_{xx}(x,y) = 2y \;\text{ or }\; f_{xx}(1,-2) = -4; \; f_{xy}(x,y) = 2x \;\text{ or }\; f_{xy}(1,-2) = 2; \; f_{yy}(x,y) = 0 \;\text{ or }\; f_{yy}(1,-2) = 0$$

$$f_{xxx}(1,-2) = 0; \; f_{xxy}(1,-2) = 2; \; f_{xyy}(1,-2) = 0; \; f_{yyy}(1,-2) = 0.$$

All partial derivatives of higher order vanish.
Taylor's expansion of $f(x,y)$ in powers of $(x-a)$ and $(y - b)$ is given by

$$f(x,y) = f(a,b) + [(x-a)f_x(a,b) + (y-b)f_y(a,b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a,b)$$

$$+2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] + \frac{1}{3!}[(x-a)^3 f_{xxx}(a,b)$$

$$+3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)] + \cdots \quad (1)$$

$$\Rightarrow \quad x^2 y + 3y - 2 = -10 + [(x-1)(-4) + (y+2)4] + \frac{1}{2}[(x-1)^2(-4) + 2(x-1)(y+2)(2) + (y+2)^2(0)]$$

$$+\frac{1}{6}[(x-1)^3(0) + 3(x-1)^2(y+2)(2) + 3(x-1)(y+2)^2(0) + (y+2)^3(0)]$$

$$= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2)$$

Example 2.33 Expand $e^x \log(1+y)$ in powers of x and y upto the terms of third degree.

Solution: Here, $f(x,y) = e^x \log(1+y)$ or $f(0,0) = 0$; $f_x(x,y) = e^x \log(1+y)$ or $f_x(0,0) = 0$

$$f_y(x,y) = e^x \frac{1}{1+y} \quad \text{or} \quad f_y(0,0) = 1; \; f_{xx}(x,y) = e^x \log(1+y) \quad \text{or} \quad f_{xx}(0,0) = 0;$$

$$f_{xy}(x,y)=e^x\frac{1}{1+y} \quad \text{or} \quad f_{xy}(0,0)=1; \; f_{yy}(x,y)=-e^x(1+y)^{-2} \quad \text{or} \quad f_{yy}(0,0)=-1;$$

$$f_{xxx}(x,y)=e^x\log(1+y) \quad \text{or} \quad f_{xxx}(0,0)=0; \; f_{xxy}(x,y)=e^x\frac{1}{1+y} \quad \text{or} \quad f_{xxy}(0,0)=1$$

$$f_{xyy}(x,y)=-e^x(1+y)^{-2} \quad \text{or} \quad f_{xyy}(0,0)=-1; \; f_{yyy}(x,y)=2e^x(1+y)^{-3} \quad \text{or} \quad f_{yyy}(0,0)=2$$

Maclaurin's expansion of $f(x, y)$ is given by

$$f(x,y)=f(0,0)+\{xf_x(0,0)+yf_y(0,0)\}+\frac{1}{2!}\{x^2f_{xx}(0,0)+2xy\;f_{xy}(0,0)+y^2f_{yy}(0,0)\}$$

$$+\frac{1}{3!}\{x^3f_{xxx}(0,0)+3x^2y\;f_{xxy}(0,0)+3xy^2f_{xyy}(0,0)+y^3f_{yyy}(0,0)\}+\cdots$$

$$\therefore \quad e^x\log(1+y)=0+x(0)+y(1)+\frac{1}{2!}\{x^2(0)+2xy(1)+y^2(-1)\}$$

$$+\frac{1}{3!}\{x^3(0)+3x^2y(1)+3xy^2(-1)+y^3(2)\}+\cdots$$

$$=y+xy-\frac{1}{2}y^2+\frac{1}{2}(x^2y-xy^2)+\frac{1}{3}y^3+\cdots$$

Example 2.34 Find the expansion of cos $x$cos y in powers of x, y up to fourth-order terms.

Solution: Let $f(x,y)=\cos x\cos y$ or $f(0,0)=\cos 0\cos 0=1$; $f_x(x,y)=-\sin x\cos y$ or $f_x(0,0)=0$

$$f_y(x,y)=-\cos x\sin y \quad \text{or} \quad f_y(0,0)=0; \; f_{xx}=-\cos x\cos y \quad \text{or} \quad f_{xx}(0,0)=-1$$

$$f_{xy}(x,y)=\sin x\sin y \quad \text{or} \quad f_{xy}(0,0)=0; \; f_{yy}(x,y)=-\cos x\cos y \quad \text{or} \quad f_{yy}(0,0)=-1$$

$$f_{xxx}(x,y)=\sin x\cos y \quad \text{or} \quad f_{xxx}(0,0)=0; \; f_{xxy}(x,y)=\cos x\sin y \quad \text{or} \quad f_{xxy}(0,0)=0$$

$$f_{xyy}(x,y)=\sin x\cos y \quad \text{or} \quad f_{xyy}(0,0)=0; \; f_{yyy}(x,y)=\cos x\sin y \quad \text{or} \quad f_{yyy}(0,0)=0$$

$$f_{xxxx}(x,y)=\cos x\cos y \quad \text{or} \quad f_{xxxx}(0,0)=1; \; f_{xxxy}(x,y)=-\sin x\sin y \quad \text{or} \quad f_{xxxy}(0,0)=0$$

$$f_{xxyy}(x,y)=\cos x\cos y \quad \text{or} \quad f_{xxyy}(0,0)=1; \; f_{xyyy}(x,y)=-\sin x\sin y \quad \text{or} \quad f_{xyyy}(0,0)=0$$

$$f_{yyyy}(x,y)=\cos x\cos y \quad \text{or} \quad f_{yyyy}(0,0)=1$$

Maclaurin's expansion of $f(x,y)$ is given by

$$f(x,y)=f(0,0)+[xf_x(0,0)+y\,f_y(0,0)]+\frac{1}{2!}[x^2\,f_{xx}(0,0)+2xy\,f_{xy}(0,0)+y^2\,f_{yy}(0,0)]$$

$$+\frac{1}{3!}[x^3f_{xxx}(0,0)+3x^2y\,f_{xxy}(0,0)+3xy^2\,f_{xyy}(0,0)+y^3f_{yyy}(0,0)]$$

$$+\frac{1}{4!}[x^4\;f_{xxxx}(0,0)+4x^3y\,f_{xxxy}(0,0)+6x^2y^2\;f_{xxyy}(0,0)+4xy^3\;f_{xyyy}(0,0)+y^4\;f_{yyyy}(0,0)]+\cdots$$

$$\therefore \quad \cos x \cos y = 1+(x\cdot 0+y\cdot 0)+\frac{1}{2!}[x^2(-1)+2xy\cdot 0+y^2(-1)]+\frac{1}{3!}(x^3\cdot 0+3x^2y\cdot 0+3xy^2\cdot 0+y^3)$$

$$+\frac{1}{4!}[x^4\cdot 1+4x^3y\cdot 0+6x^2y^2\cdot 1+4xy^3\cdot 0+y^4\cdot 1]+\cdots$$

$$=1-\frac{x^2}{2!}-\frac{y^2}{2!}+\frac{x^4}{4!}+\frac{6x^2y^2}{4!}+\frac{1}{4!}y^4+\cdots$$

$$=1-\frac{1}{2}(x^2+y^2)+\frac{1}{24}(x^4+6x^2y^2+y^4)+\cdots$$

Example 2.35 Expand $x^3 + y^3 + xy^2$ in the power of $x - 1$ & $y - 2$ upto 3 degree terms .

Solutions:

Let $f(x, y) = x^3 + y^3 + xy^2$

Let $f(x, y) = x^3 + y^3 + xy^2$ $f(1, 2) = 1 + 8 + 4 = 13$

$f_x(x, y) = 3x^2 + y^2$ $\quad f_x(1, 2) = 3 + 4 = 7$

$f_y(x, y) = 3y^2 + 2xy$ $\quad f_y(1, 2) = 12 + 4 = 16$

$f_{xx}(x, y) = 6x$ $\quad f_{xx}(1, 2) = 6$

$f_{xy}(x, y) = 2y$ $\quad f_{xy}(1, 2) = 4$

$f_{yy}(x, y) = 6y + 2x$ $\quad f_{yy}(1, 2) = 6 + 4 = 10$

$f_{xxx}(x, y) = 6$ $\quad f_{xxx}(x, y) = 6$

$f_{xxy}(x, y) = 0$ $\quad f_{xxy}(x, y) = 0$

$f_{xyy}(x, y) = 2$ $\quad f_{xyy}(x, y) = 2$

$f_{yyy}(x, y) = 6$ $\quad f_{yyy}(x, y) = 6$

$$f(x,y)=13+\left[(x-1)7+(y-2)16\right]+\frac{1}{2!}\left[6(x-1)^2+8(x-1)(y-2)+14(y-2)^2\right]+$$

$$\frac{1}{3!}\left[6(x-1)^3+0+6\left(x-1\right)\left(y-2\right)+6(y-2)^3\right]$$

$$f(x,y)=13+\left[7x-7+16y-32\right]+\left[3(x-1)^2+4(x-1)(y-2)+7(y-1)^2\right]+\left[(x-1)^3+(x-1)(y-2)+(y-2)^3\right]$$

Example 2.36 Expand the function sinxy in power of $x - 1$ and $y-\frac{\pi}{2}$ upto second degree terms.

Solutions: Let $f(x, y) = \sin xy$ $\quad f\left(1,\frac{\pi}{2}\right)= 1$

$f_x(x,y)=y\cos(xy)$ $\quad f_x\left(1,\frac{\pi}{2}\right)=0$

$$f_y(x,y) = x\cos(xy) \qquad f_y\left(1,\frac{\pi}{2}\right) = 0$$

$$f_{xx}(x,y) = -y^2\sin(xy) \qquad f_{xx}\left(1,\frac{\pi}{2}\right) = -\frac{\pi^2}{4}$$

$$f_{yy}(x,y) = -x^2\sin(xy) \qquad f_{yy}\left(1,\frac{\pi}{2}\right) = -1$$

$$f_{xy}(x,y) = -xy\sin(xy) + \cos(xy) \qquad f_{xx}\left(1,\frac{\pi}{2}\right) = \frac{\pi}{2}$$

$$= \sin xy\ 1 + \left[(x-1)0 + \left(y-\frac{\pi}{2}\right)0\right] + \frac{1}{2!}\left[(x-1)\left(y-\frac{\pi}{2}\right)\left(-\frac{\pi}{2}\right) + \left(y-\frac{\pi}{2}\right)^2(-1)\right]$$

$$= 1 + \frac{1}{2}\left[\frac{-\pi^2}{4}(x-1)^2 - \pi(x-1)\left(y-\frac{\pi}{2}\right) - \left(y-\frac{\pi}{2}\right)^2\right]$$

Example 2.37 Expand $e^x\cos y$ about $\left(0,\frac{\pi}{2}\right)$ upto the third term using Taylor's series.

Solutions: Let $f(x, y) = e^x\cos y$

$$f_x(x,y) = e^x\cos y \qquad f_x\left(0,\frac{\pi}{2}\right) = 0$$

$$f_y(x,y) = -e^x\sin y \qquad f_y\left(0,\frac{\pi}{2}\right) = -1$$

$$f_{xx}(x,y) = e^x\cos y \qquad f_{xx}\left(0,\frac{\pi}{2}\right) = 0$$

$$f_{yy}(x,y) = -e^x\cos y \qquad f_{yy}\left(0,\frac{\pi}{2}\right) = 0$$

$$f_{xxx}(x,y) = e^x\cos y \qquad f_{xxx}\left(0,\frac{\pi}{2}\right) = 0$$

$$f_{xxy}(x,y) = -e^x\sin y \qquad f_{xxy}\left(0,\frac{\pi}{2}\right) = -1$$

$$f_{xyy}(x,y) = -e^x\sin y \qquad f_{xyy}\left(0,\frac{\pi}{2}\right) = -0$$

$$f_{yyy}(x,y) = e^x\sin y \qquad f_{yyy}\left(0,\frac{\pi}{2}\right) = 1$$

$$e^x\cos y = 1 + \frac{x}{1!} + \frac{1}{2!}(x^2 - y^2) + \frac{1}{3!}(x^3 - 3xy^2) + \ldots$$